Bipartite Subgraphs of Large Odd Girth Series-Parallel Graphs

Tomás Feder* Carlos Subi†

Abstract

We consider unweighted and weighted versions of the minimum vertex deletion bipartition problem on n-vertex series-parallel graphs of odd girth g. One cannot hope to prove a bound better than $\lfloor \frac{n}{g} \rfloor$ on the weight of vertices S to be deleted to obtain a bipartite graph, even for the unweighted subcubic version. We obtain a ratio of $\frac{1}{g-c}$ for various versions of the problem, with $c = 0, 1, 2$ along the lines of G subcubic or not, weighted or not, and S independent or not. One of these results is improved by the Kneser graph technique described below.

We also show that the minimum edge deletion bipartition problem has a tight bound $\lfloor \frac{m}{g} \rfloor$ on the number of edges to be deleted to obtain a bipartite graph, on m-edge series-parallel graph of odd girth g, and extend this result to the weighted case. This last result is based on mapping G to the Kneser graph $K(g, h)$ for $g = 2h + 1$.

1 Introduction

We consider series-parallel graphs as special cases of graphs that do not contain a K_4 minor. Such graphs can be repeatedly decomposed by considering components in series and in parallel. A *subcubic* graph is a graph with vertices of degrees at most three. The *odd girth* of a graph is the length of the shortest odd cycle. We note that series-parallel graphs, and more generally graphs without a K_4 minor, can be 3-colored.

We consider unweighted and weighted versions of the minimum vertex deletion bipartition problem on n-vertex series-parallel graphs G of odd girth g. One cannot hope to prove a bound better than $\lfloor \frac{n}{g} \rfloor$ on the weight of the set S of vertices to be deleted to obtain a bipartite graph, even for the unweighted subcubic version. We obtain (1) a ratio of $\frac{1}{g-2}$ for the weighted case where S is independent, (2) a ratio of $\frac{1}{g-1}$ for the weighted case with S not necessarily independent, and a ratio of $\frac{1}{g}$ when (3) G is subcubic and weighted with S independent, or when (4) G is unweighted and S is not necessarily independent.

We show that the fractional chromatic number of a series-parallel graph G (or a graph without K_4 minor) of odd girth $g = 2h + 1$ is exactly $\frac{g}{h}$. The lower bound follows from the fact that G contains a cycle of length g, with maximum independent number h. The upper bound is obtained by constructing a homomorphism from G to the Kneser graph $K(g, h)$. In particular, the graph $K(5, 2)$ is the Petersen graph, giving fractional chromatic number at most $\frac{5}{2}$ to triangle-free series-parallel graphs G. This result implies that given an edge-weighted series-parallel graph (with nonnegative weights) of odd girth g, we can remove edges of weight at most $\frac{1}{g}$ of the total weight and obtain a bipartite graph (in fact the removed edges themselves form a bipartite subgraph). The Kneser result also implies that the minimum weight vertex deletion problem has an independent

*268 Waverley Street, Palo Alto, CA 94301. Email: tomas@theory.stanford.edu
†carlos.subi@hotmail.com

set solution of weight at most $\frac{1}{4} + \frac{1}{4g}$ of the total weight. In particular, triangle-free graphs ($g = 5$) give a ratio of $\frac{3}{10} < \frac{1}{3}$.

Subcubic triangle-free graphs have fractional chromatic number at most $\frac{14}{5}$ [2], matching a lower bound over the entire class [3] as this class contains graphs with maximum independent number $\frac{5}{14}n$. Triangle-free subcubic planar graphs contain an independent set of size at least $\frac{3}{8}n$ [4], and the constant $\frac{3}{8}$ is best possible over this class.

Every 2-connected subcubic triangle-free graph contains an induced bipartite subgraph of order at least $\frac{5}{7}n$ [7], except for the Peterson graph and the dodecahedron. Our bound of $\frac{g-1}{g}n$ improves this bound in the series-parallel case.

Large odd girth series-parallel graphs were also studied in the context of the circular chromatic number. That approach only gives bound of $\frac{3}{2g+c}$ for the minimum vertex deletion bipartition problem, based on homomorphisms to large odd cycles [1, 5, 6].

A *series-parallel* graph is a component C with endpoints x, y (sometimes called source and sink) that is either (1) a single edge joining x, y, or (2) two components C_1, C_2 with endpoints x, y, the parallel case, or (3) two components C_1, C_2 with endpoints x, z, and z, y, the series case. The components C_1, C_2 must in turn be series-parallel as well. A graph without K_4 minor is also a graph such that each biconnected component is series-parallel. A *homomorphims* f from a graph G to a graph H maps vertices to vertices so that if $xy \in E(G)$ then $f(x)f(y) \in E(H)$.

2 The Weighted Case and Fractional Colorings

The Kneser graph $K(g, h)$ for $1 \leq h \leq g$ is the graph whose vertices are the subsets of size h of a base set $\{1, 2, \dots, g\}$ of size g, and whose edges join two subsets if they are disjoint. Given a graph G, we denote by $d_e^G(v, w)$ and $d_o^G(v, w)$ the lengths of the shortest even and odd paths joining two vertices v, w in G respectively. The fractional chromatic number of a graph G is the infimum of $\frac{g}{h}$ such that G maps by a homomorphism to $K(g, h)$. Equivalently, if we assign nonnegative weights w_i to the vertices v_i so that the total weight of G is $w(G)$ and a maximum weight independent set in G has weight $w(I)$, then the minimum over all weight assignments to G of $\frac{w(I)}{w(G)}$ is $\frac{h}{g}$.

Lemma 1 *Let $H = K(g, h)$ for $g = 2h + 1$. Given two vertices v, w in H with $|v \cap w| = d$, we have $d_e^H(v, w) = 2(h - d)$ and $d_o^H(v, w) = 2d + 1$, with $d_e^H(v, w) + d_o^H(v, w) = g$.*

Proof. This is obtained by considering an induced odd cycle of girth g on the vertices $\{1 + c, 2 + c, \dots, h + c\}$ where addition is modulo g and adding one to c gives two vertices at distance two in H. □

Lemma 2 *Suppose integers $0 \leq d_1 \leq d_2 \leq d_3 \leq h$ with*

(1) $(2d_1 + 1) + (2d_2 + 1) + (2d_3 + 1) \geq g$ *and*

(2) $(2d_1 + 1) + 2(h - d_2) + 2(h - d_3) \geq g$.

Then there exist vertices u, v, w in $H = K(g, h)$ for $g = 2h + 1$ such that $|u \cap v| = d_1, |u \cap w| = d_2$ and $|v \cap w| = d_3$.

Proof. We shall choose $d = |u \cap v \cap w|$ by letting $d = 0$ if $d_2 + d_3 \leq h$ and $d = d_2 + d_3 - h$ if $d_2 + d_3 > h$. We only need to satisfy the inequalities

(A) $d \leq d_1$

(B) $d_2 + d_3 - d \leq h$ and

(C) $3h - d_1 - d_2 - d_3 + d \leq g$

by the inclusion-exclusion principle. In the first case where $d = 0$, we have (A) since $d = 0 \leq d_1$, (B) since $d_2 + d_3 \leq h$, and (C) since (1) gives $d_1 + d_2 + d_3 \geq \frac{g-3}{2} = h - 1 + d$ satisfied by (1). In the second case where $d = d_2 + d_3 - h$, we have (A) since (2) gives $d = d_2 + d_3 - h \leq h - \frac{g-1}{2} + d_1 = d_1$, (B) since $d_2 + d_3 - d = h$, and (C) since $3h - d_1 - d_2 - d_3 + d = 2h - d_1 \leq g$. $\square$

We consider a series-parallel graph G of odd girth at least $g = 2h + 1$. First we transform G by adding to each edge u, v a parallel path from u to v of length $g - 1$. More generally, any two vertices of G that are adjacent now joined by two paths, one of some length $1 \leq d \leq g - 1$ and the other of length $g - d$. These two paths form a *super-edge*.

Theorem 1 *A series-parallel (K_4 minor free) graph G of odd girth at least $g = 2h + 1$ can be mapped by a homomorphism to $H = K(g, h)$.*

Proof. Consider a series-parallel graph G whose edges are super-edges. G must have a vertex v of degree 2, with super-edges uv, vw. We add a super-edge uw with even and odd lengths d_e, d_o chosen such that $d_e + d_o = g$, $d_e + d_o^G(u, w) \geq g$, and $d_o + d_e^G(u, w) \geq g$. This is feasible since $d_e^G(u, w) + d_o^G(u, w) \geq g$.

We remove then the vertex v and its two super-edges, proceed by induction on the smaller graph, and add v again since the triangle is feasible by Lemmas 1 and 2. To extend the result to G with forbidden minor K_4, it suffices to notice that each biconnected component of G is series-parallel and H is vertex transitive, so mapping each biconnected component suffices. $\square$

Corollary 1 *A series-parallel (K_4 minor free) graph G of odd girth $g = 2h + 1$ has fractional chromatic number $\frac{g}{h}$.*

Proof. The upper bound follows from Theorem 1, and the lower bound by considering the maximum weight independent set when a cycle of length g in G has weight 1 at each vertex is h. $\square$

Corollary 2 *The minimum edge deletion bipartition weight for a series-parallel (K_4 minor free) graph G of odd girth g with weights on the edges is at most $\frac{1}{g}$ of the total edge weight (and the edges deleted may be assumed to form a bipartite graph and contain exactly one edge from each cycle of length g).*

Proof. Each edge in $K(g, h)$ for $g = 2h + 1$ omits one $1 \leq i \leq g$. Removing the edges mapping to omitting i with least weight gives the bound $\frac{1}{g}$, and we are left with the bipartite graph with the vertices omitting this i in one side and the vertices including this i in the other side. The edges removed for a given i do not share endpoints nor cycles of length g in $K(g, h)$, and therefore form a bipartite graph such that no two edges removed belong to a cycle of length g in G. $\square$

This result is tight even in the unweighted case by considering a path of cycles of length g attached at vertices, giving for all m-edge series-parallel graphs the bound $\lfloor \frac{m}{g} \rfloor$.

Corollary 3 *The minimum weight of an independent set I of a series-parallel (K_4 minor free) graph G of odd girth g with weights on the vertices such that $G - I$ is bipartite, is at most $\frac{1}{4} + \frac{1}{4g}$ of the total edge weight. In particular, for triangle-free graphs ($g = 5$), a ratio of $\frac{3}{10}$ is achievable.*

Proof. We choose the bipartite graph to have in one side the vertices containing the $1 \leq i \leq g$ giving the largest weight, at least $\frac{h}{g}$ of the total (since the union of all g of them counts each vertex h times), and on the other side at least half of the remaining vertex weights, by choosing one of each pair of vertices that are adjacent and omit i. The bound achieved is then at most $\frac{h+1}{2g} = \frac{1}{4} + \frac{1}{4g}$ as desired. □

3 The Unweighted Case

The following result states that removing a small color class of 3 colors in a subcubic graph renders a large induced bipartite subgraph.

Theorem 2 *A series-parallel (K_4 minor free) subcubic graph G of odd girth $g = 2h + 1$ with n vertices can be 3-colored so that some color class L contains at most $\lfloor n/g \rfloor$ vertices.*

Proof. Let G be a series-parallel graph from source s to sink t. Repeatedly decomposing G into series and parallel components, we eventually reach a component C containing an odd cycle, with C consisting of two parallel components that do not contain an odd cycle. We shall eventually remove one of the two endpoints of C to break all the odd cycles within C. For now, we avoid paths going through C and proceed to find more such components $C', C'', \ldots$, until all odd cycles are broken. Note that each component C detected has at least g vertices, and only one of these vertices is removed, giving the bound $\frac{n}{g}$ on vertices removed.

It remains to choose the vertices to so remove so that these vertices form an independent set. The only problem is that there may be chains of components $C, C', C'', \ldots$, such that the top vertex of one component is adjacent to the bottom vertex of the next component. In fact, we may even have a cycle of such components if we go through s and t. For a chain, we may remove always the bottom vertex, and when going through s or t, switch from bottom to top vertex.

To generalize to the K_4 minor free case, solve each biconnected component separately, and two biconnected components are joined by an edge at their vertices of degree 2, which were never chosen to be removed in this construction except in the case where the biconnected component is a cycle. For this last case, we proceed from leaf biconnected components up, and always choose a vertex in a cycle component not connected to the parent. This preserves independence of removed vertices. □

Note that the bound is tight for G a cycle of length $g = 2h+1$, and for all n as well by attaching such cycles with joining edges or paths.

The proof extends as follows by assuming the girth is g, not just the odd girth.

Corollary 4 *A series-parallel (K_4 minor free) subcubic graph G of girth g with n vertices can be 3-colored so that some color class L contains at most $\lfloor n/g \rfloor$ vertices, and the other two classes together induce a forest.*

This last problem has a lower bound for approximation of $\frac{1}{\lceil \frac{g}{2} \rceil + 1}$ in the weighted case. To see this, join two vertices x, y with three disjoint paths of length $\lceil \frac{g}{2} \rceil$, give weight 3 to x, y, and weight 1 to the remaining vertices.

Theorem 3 *A series-parallel (K_4 minor free) graph G of odd-girth $g = 2h + 1$ with n vertices has a (not necessarily independent) set of at most $\lfloor \frac{n}{g} \rfloor$ vertices whose removal produces an induced bipartite graph.*

Proof. We proceed by induction on G. Suppose G is series-parallel. As in the subcubic case, identify the components $C, C', C'', \ldots$ to remove an endpoint from each. This time such components may share endpoints, so the count must be done carefully. If C and C' share the endpoint v, we remove v and count v together with the internal vertices, for a total of at least $(g-1)+1 = g$. The remaining components share no endpoints, so we count all their vertices, at least g. In this last case, we must make sure that no larger component C' has a path not going inside C but through the endpoint of C not chosen. But this can be a problem only for one of the two endpoints, so we also always choose the vertex belonging to the outmost component in the series-parallel decomposition.

Suppose instead G is without K_4 minor but not biconnected. Decompose G into S and T joined at a common vertex x, and of sizes s and t. If s is not a multiple of g, then $\lfloor \frac{s}{g} \rfloor + \lfloor \frac{t}{g} \rfloor = \lfloor \frac{s-1}{g} \rfloor + \lfloor \frac{t}{g} \rfloor \leq \lfloor \frac{n}{g} \rfloor$ and the result follows by induction on the two components S and T. If s is a multiple of g, then remove x and solve $S - x$ and $T - x$ separately. The result again follows by induction since $1 + \lfloor \frac{s-1}{g} \rfloor + \lfloor \frac{t-1}{g} \rfloor = \lfloor \frac{s}{g} \rfloor + \lfloor \frac{t-1}{g} \rfloor = \lfloor \frac{n}{g} \rfloor$. □

The proof extends as follows by assuming the girth is g, not just the odd girth.

Corollary 5 *A series-parallel (K_4 minor free) graph G of girth g with n vertices has a (not necessarily independent) set of at most $\lfloor \frac{n}{g} \rfloor$ vertices whose removal produces an induced forest.*

This last problem has a lower bound for approximation of $\frac{1}{\lceil \frac{g}{2} \rceil + 1}$ in the weighted case as described before in the subcubic case. An upper bound of $\frac{1}{\lceil \frac{g}{2} \rceil - 1}$ in the weighted case can be achieved as follows. Find a component C consisting of a cycle joined at two vertices x, y; one of the two paths joining x, y has length at least $\lceil \frac{g}{2} \rceil$, and we may remove this path p after selecting to remove the least weight internal vertex on p.

4 The Weighted Case

We now study the weighted version of the problem.

Theorem 4 *Let G be a cubic series-parallel (K_4 minor free) graph of odd girth g and with nonnegative weights on the vertices. Then G has an independent set S whose sum weight is at most $\frac{1}{g}$ of the total weight such that $G - S$ is bipartite.*

Proof. Assume first that G is biconnected, thus series-parallel. Orient the edges from the source s to the sink t.

Consider components C entered at x and exited at y, such that x has in-degree one and out-degree two, and y has in-degree two and out-degree one. We shall first simplify the algorithm by assuming that x and y have the same weight, otherwise replace their weights by the average of the two weights.

We shall simplify further each C by making tentative choices as to which vertices might be included in S, so C will consist of two paths from x to y, which will get simplified to just one. We shall also introduce the notion of an edge of infinite length.

So suppose C consists of two paths p and q, which we wish to replace by a single path. Suppose p, q have lengths ℓ_p, ℓ_q and minimum internal weights w_p, w_q. If say $\ell_p = 1$ then w_p is infinite. If ℓ_p, ℓ_q are both odd or both even, and say $\ell_p \leq \ell_q$, then we replace C by a path of length ℓ_p and internal weights $w_p + w_q$. The weights of x, y are left the same.

If ℓ_p, ℓ_q are one odd and one even, so that $\ell_p + \ell_q \geq g$ is odd, then let w be the minimum weight on C, and subtract w from the weight of every vertex in C. This subtraction of weight at least gw from the total weight justifies selecting a vertex z of weight w for S. If z can be x or y (which have the same weight by assumption), then replace C by an edge of infinite length. Otherwise z can be chosen say in p, and C is replaced by q.

If say p contains an infinite length edge, then replace C by q. More generally, it may be in the induction that choosing z from p just changes the parity of p.

Suppose at some stage we chose z of weight w for S. If z was chosen from a C with both ℓ_p, ℓ_q of the same parity, then we choose w_p from p and w_q from q out of $w_p + w_q$. If z was chosen from C with one even and one odd, then we proceed to assign the remaining w in ℓ_q, so at this point both of p, q contain a zero weight, but only the element in q will now be put in S. This is a switch of parity that propagates down the subcomponents. If at any point we choose one of the endpoints x, y of same weight, we do not designate which one yet.

If a vertex of degree two was chosen for S, and later a neighbor of degree three is chosen for S, then we de-choose the vertex of degree two. We show that the vertices of degree three can be chosen independent. These are the vertices x, y of some component C. If x (y) was chosen, choose out-vertices (in-vertices) of components within p, q. This actually gives two disjoint choices (x or y for each C) of independent set. Choose out of the two choices the one of least total weight in the original weight assignment. The total chosen weight will be at most to weight corresponding to choosing one of x and y for each C, which is half the total weight.

In the end we are left with a single component C with $x = s$ and $y = t$, and after simplifying C to a single path from s to t we are done.

If G without K_4 minor is not biconnected, we may consider a leaf component T joined to R via an edge tr. If the vertex t was not chosen in solving the series-parallel graph T with sink t, then examine the final cycle component C, with $x = s$ and $y = t$, and add the weight of s to r before solving R. If when solving R the vertex r is chosen, choose s as well to break C; and if r is not chosen, choose t of weight zero. $\square$

Theorem 5 *Let G be a series-parallel (K_4 minor free) graph of odd girth g and with nonnegative weights on the vertices. Then G has a (not necessarily independent) S whose sum weight is at most $\frac{1}{g-1}$ of the total weight such that $G - S$ is bipartite.*

Proof. We proceed as in Theorem 4 but subtract w from vertices of C other than one of x, y only. The idea is that we may end up choosing both x and y, but only one is charged for C.

If G has no K_4 minor and has two components T and R joined at x, we identify an edge tx in T with an edge rx in R and add the weights of t and r. Enough such identifications will make G biconnected and thus series-parallel. $\square$

Theorem 6 *Let G be a series-parallel (K_4 minor free) graph of odd girth g and with nonnegative weights on the vertices. Then G has an independent S whose sum weight is at most $\frac{1}{g-2}$ of the total weight such that $G - S$ is bipartite.*

Proof. We proceed as in Theorem 5 but subtract w from vertices of C other than x, y. The idea is that if we choose say $z = x$, then we need not choose any neighbor of z in a smaller component of the series-parallel decomposition, to guarantee independence. We proceed in this manner from outer to inner components of the series-parallel decomposition. □

Define a series-parallel G to be *cyclic* if it consists of components C like the one joining x, y in the above proof joined in a cycle. As in the above proof, each C may be simplified either to a single path or to two paths p, q with lengths of different parity (the simplification does not work further because of degree two both in and out of x and y).

Theorem 7 *Cyclic graphs G also allow finding an independent S whose sum weight is at most $\frac{1}{g}$ of the total weight of vertices, and such that $G - S$ is bipartite.*

Proof. Denote the cycle $x_0, C_1, x_1, C_2, x_2, \ldots, x_{k-1}, C_k, x_k = x_0$, where C_i is a component with endpoints x_{i-1}, x_i. Let w_i be the least internal weight in C_i, unless C_i is just a path, in which case $w_i = 0$. Let v_i be the weight of x_i. Let $W = \sum_i w_i$ and $V = \sum_i v_i$. Let $U = 2W - V$.

Suppose $U \leq 0$. Then $\sum_i w_i + w_{i+1} - v_i \leq 0$. Proceed as in the previous proof to subtract the minimum weight w_i from each element of C_i, or subtract as much as possible until one of v_{i-1}, v_i becomes zero. The rest is subtracted from other v_i. This will only happen if $w_i + wi + 1 - v_i > 0$ for some i. In this case include in S this x_i and all the corresponding vertices of weight w_j from C_j, $j \neq i, i+1$. Otherwise all $w_i + w_{i+1} - v_i \leq 0$, and after the subtractions, simplify all C_i to paths as in the previous proof, so G becomes a cycle L. If L is of odd length at least g, remove the element of L of least weight.

Suppose instead $U > 0$. Then $\sum_i w_i + w_{i+1} - v_i > 0$. View G as a cycle of length k, and 3-color the vertices x_i (2-color if k is even). Let the three color classes be K_j. For each K_j, choose x_i for S if $x_i \in K_j$, and add the element of weight w_i from C_i to S if $x_{i-1}, x_i \notin K_j$. The set S will then have total weight $(\sum_i w_i) - (\sum_{x_i \in K_j} w_i + w_{i+1} - v_i)$. Averaging over all three K_j, one of the three colors gives weight at most $(\sum_i w_i) - \frac{1}{3}(\sum_i w_i + w_j - v_i) = W - \frac{U}{3}$ The total weight is at least $gW - U \geq g(W - \frac{U}{3})$ since $g \geq 3$. □

5 The Weighted Case and Circular Colorings

The following Theorem is known.

Theorem 8 *[6] A series-parallel (K_4 minor free) graph G of odd girth g admits a homomorphism to an odd cycle of lenght ℓ, where $\ell = 4k+1$ for $g = 6k+1$, and $\ell = 4k+3$ for $g = 6k+3$. If $g = 6k-1$, then G admits a homomorphism to the graph with $8k$ vertices, integers modulo $8k$, such that two vertices v, w are adjacent if $4k-1 \leq |v-w| \leq 4k+1$.*

Corollary 6 *The minimum weight of an independent set I of a series-parallel (K_4 minor free) graph G of odd girth g with weights on the vertices such that $G - I$ is bipartite, is at most $\frac{3}{2g+c}$ of the total edge weight, where the constant $c \geq 0$ $c = 1$ for $g = 6k+1$, $c = 3$ for $g = 6k+3$, and $c = -1$ for $g = 6k+5$.*

Proof. The bound follows from Theorem 8. We may chose the vertex on the cycle of length ℓ of least weight preimage, which gives us an independent set of weight at most $\frac{1}{\ell}$ of the total weight whose removal leaves us with an induced bipartite graph of weight at least $\frac{\ell-1}{\ell}$ of the total weight (the preimage of a path of length $\ell - 1$). $\square$

The following definition is from [6]. Given two integer k, d such that $k \geq d$, a (k,d)-coloring of a graph G is a coloring c of the vertices of G with colors $1, 2, \ldots, k$ such that for any two adjacent vertices x and y of G we have $d \leq |c(x) - c(y)| \leq k - d$. The *circular chromatic number* of G is define as the infimum of the ratio k/d for which there exissts a (k,d)-coloring of G. If G is finite then the infimum is always attained. For every graph containing at least one edge, the circular chromatic number of G is at least 2, and is exactly 2 if G is bipartite. If G is series-parallel, then either G contains a triangle, in which case its circular chromatic number is 3, or G is triangle-free, in which case its circular chromatic number is between 2 and $\frac{8}{3}$. Theorem 8 gives tight bounds on the circular chromatic number achievable for all graphs of odd girth g [6].

Theorem 9 *Suppose a vertex-weighted graph has a (k,d)-coloring achieving circular chromatic number $2 \leq r = \frac{k}{d} \leq 3$. Then we may remove an independent set I whose sum weight is a fraction at most $1 - \frac{2}{r} = 1 - \frac{2d}{k}$ of the total weight and obtain a bipartite subgraph.*

Proof. Consider the k independent sets $I_i = \{1+i, 2+i, \ldots, k-2d+i\}$ for $0 \leq i \leq k-1$ modulo k. These sets are independent since $k - 2d - 1 < d$. If we superpose these k sets I_i, each element is counted $k - 2d$ times, so some such I_i must receive at most $\frac{k-2d}{k} = 1 - \frac{2}{r}$ of the total weight, say I_d. Removing I_d leaves us with a bipartite subgraph $B = (\{1, \ldots, d\}, \{k-d+1, \ldots, k\}, E)$ which is the image of a bipartite subgraph of G. $\square$

The following result shows that homomorphisms to cycles closely approximate circular chromatic number.

Theorem 10 *Suppose $2 < \frac{k}{d} \leq 3$. If G is (k,d)-colorable, then G has a homomorphism to an r-cycle for odd $3 \leq r \leq 2\lfloor \frac{d}{k-2d} \rfloor + 1$. If G has a homomorphism to an r-cycle with r odd, then G is (k,d)-colorable for $3 \leq r \leq 2\lceil \frac{d}{k-2d} \rceil + 1$.*

Proof. Suppose G is (k,d)-colorable. Let $s = \lfloor \frac{d}{k-2d} \rfloor \geq 1$, and let s' be d modulo $k - 2d$. Define $2s + 1$ intervals partitioning the integers in $[1,k]$ by letting $I_0 = [1, k - 2d + s'], I_i = [i(k-2d) + s' + 1, (i+1)(k-2d) + s']$ for $1 \leq i \leq s$, $I_{s+1} = [k-d+1, 2k-3d+s'], I_{s+1+i} = [k - d + s' + 1 + i(k-2d), k - d + s' + (i+1)(k-2d)]$ for $1 \leq i \leq s-1$. Mapping I_i to $i+1$ gives a $(2s+1, s)$-coloring of G, that is, a homomorphism from G to the cycle of length $2s+1$, which can in turn by mapped to the r-cycle for odd $3 \leq r \leq 2s+1$.

Suppose instead G has a homomorphism to the r-cycle for $r = 2t + 1$ for $t = \lceil \frac{d}{k-2d} \rceil \geq 1$. Write the r-cycle as $v_t, v_{t-1}, \ldots, v_1, u, w_1, w_2, \ldots, w_t$. First assign a color to u, then greedily to the pairs v_i, w_i, such that $|c(v_i) - c(w_i)| = i(k-2d)$. Then $d \leq |c(v_t) - c(w_t)| \leq k - d$ and v_t, w_t are adjacent, completin the (k,d)-coloring of the r-cycle. $\square$

Theorem 11 *Every subcubic series-parallel graph of odd girth $g \geq 3$. has a homomorphism to the g-cycle if and only if $g = 3, 5, 7, 9$.*

Proof. The counterexample for $g \geq 11$ is given in Theorem 12.

Suppose we wish to map G of odd girth $g \leq 9$ to the g-cycle. Decompose G into series-parallel components until some component C joined at x and y is not bipartite, but contains two parallel components that are each bipartite, one giving odd distance d_o and the other one giving even distance d_e between x and y, where $d_o + d_e \geq g$. We may then simplify C to two paths of length d_o and d_e joining x and y. It is possible to reduce d_o, d_e until $d_o + d_e = g$ without violating the girth g constraint.

Suppose $d_o = 1$. Then we may remove the path of length d_e. Suppose $d_e = 2$. Then we may remove the path of length d_o, since this only additionally allows mapping x and y to the same vertex, and if the incoming edges are $x'x$ and yy' this still only allows x', y' at distances $0, 2, 4$. Suppose $d_o = 3$. Then we may remove the path of length d_e. since this only additionally allows mapping x and y to distance one, and if the incoming edges are $x'x$ and yy' this still only allows x', y' at distances $0, 2, 4$.

This settles the cases $g = 5, 7$, since these have $d_o = 1, 3$ or $d_e = 2$ because $d_o + d_e = g$. For $g = 9$, we must then have $d_e = 4$ and $d_o = 5$. The vertices x', y' can then map to any distance other than $0, 1$. Consider a component C such that C has two components in parallel C_1, C_2 that do not decompose further in parallel other than via a component of type x, y. Adding one edge or one component of type x, y to the path joining $x'y'$ gives x', y'' allowing all distances but 0, and this can be simplified to a path of length 7. We may thus assume C_1 is just the component joining x', y', and since we may assume C_2 different from C_1, we have that C_2 is a path of length $2 \leq \ell \leq 5$ joining x', y'. ($\ell = 1$ violates girth, and $\ell \geq 6$ maps C_2 to C_1

Extend C with edges x'', x' and y', y''. If $\ell = 2$, then x'', y'' can go distance $0, 2, 4$ which can be represented by a path of length 4. If $\ell = 3$, then x'', y'' can go distance $1, 3, 5$ which can be represented by a path of length 5. If $\ell = 4$, then x'', y'' can go distance $0, 2, 4, 6$ which can be represented by a path of length 6. If $\ell = 5$, then x'', y'' can go distance $1, 3, 5, 7$ which can be represented by a path of length 7.

This completes the induction that reduces mapping G to succesively smaller graphs until we are left with a cycle of length $g = 9$. □

The following follows as in Corollary 6.

Corollary 7 *The minimum weight of an independent set I of a subcubic series-parallel (K_4 minor free) graph G of odd girth $g = 3, 5, 7, 9$ with weights on the vertices such that $G - I$ is bipartite, is at most $\frac{1}{g}$ of the total edge weight.*

Theorem 12 *There exists a subcubic series-parallel graph G of odd girth $g = 2h + 1 \geq 11$ such that G does not have a homomorphism to the g'-cycle for any $g' > \frac{3g+7}{4}$ if h is odd, or for any $g' > \frac{3g+5}{4}$ if h is even.*

Proof. Let $2d = g - g'$. Let G be the 6-cycle $v_1, w_1, v_2, w_2, v_3, w_3$, where the $v_i w_i$ are super-edges joining the endpoints by two paths of lengths h and $h + 1$ respectively, and $w_1 v_2, w_2 v_3, w_3 v_1$ are regular edges.

Suppose h is odd. Then the even distance d_1 of the images of v_1, v_2 is at most $h + 1$ and at least $h - 1 - 2d$, and similarly d_2 for v_2, v_3 and d_3 for v_3, v_1. Say $d_1 \geq d_2 \geq d_3$. Then we must have $d_1 = d_2 + d_3$, which is not possible if $h + 1 < 2(h - 1 - 2d)$, or $2d < \frac{h-3}{2}$, or $g' = g - 2d > 2h + 1 - \frac{h-3}{2} = \frac{3g+7}{4}$.

Suppose h is even. Then the even distance d_1 of the images of v_1, v_2 is at most $h+2$ and at least $h-2d$, and similarly d_2 for v_2, v_3 and d_3 for v_3, v_1. Say $d_1 \geq d_2 \geq d_3$. Then we must have $d_1 = d_2 + d_3$, which is not possible if $h + 2 < 2(h - 2d)$, or $2d < \frac{h-2}{2}$, or $g' = g - 2d > 2h + 1 - \frac{h-2}{2} = \frac{3g+5}{4}$. □

Theorem 13 *There exists a constant c such that for all odd $g \geq 11$ there exists a subcubic series-parallel graph G of odd girth g such that G does not have a homomorphism to the g'-cycle for any $g' > \frac{5g+c}{7}$.*

Proof. Let $g = 2h+1$, $h = 14k+40$, $g' = g - \frac{h}{2} - k = 20k+61$, with $g = 28k+81$. Let G consist of a triangle $v_1v_2v_3$ with three super-edges having parallel paths of lengths $h-2k-12$ and $h+2k+3$. In addition, G has three paths $v_iw_iv_{i+1}$ modulo 3 with two super-edges having parallel paths of lengths h and $h+1$.

This construction gives vertices of degree 4 at w_i, which we replace by a path of length 2 centered at w_i, and degree 8 at v_i, which we replace by a complete binary tree of depth 2 (4 leaves) rooted at v_i, to make G subcubic.

If all two even super-edges of the path $v_iw_iv_{i+1}$ go in the same direction, they will add up to at least $2(\frac{h}{2}-k)-6 = h-2k-6$, which is not possible since the super-edge v_iv_{i+1} gives length at most $h-2k-8$. If the two even super-edges of the path $v_iw_iv_{i+1}$ go in different direction, they will add up to at most $\frac{h}{2}+k+6$. If the super-edges v_iv_i give even lengths d_i with $d_1 \geq d_2 \geq d_3$, we must have $d_1 = d_2 + d_3$ and $d_i \geq h-2k-12-\frac{h}{2}-k-4 = \frac{h}{2}-3k-16$. Therefore $d_1 \geq h-6k-32 \leq \frac{h}{2}+k+6$, or $h \leq 14k+38$, contrary to assumption. □

Theorem 14 *There exists a constant c such that for all $g \geq 11$ there exists a subcubic series-parallel graph G of girth g such that G does not have a homomorphism to the g'-cycle for any $g' > \frac{2g-c\log g}{3}$.*

Proof. It is constructed in [1] a graph G of girth $g \geq 2\lfloor(3k-1)/2\rfloor$ that is not $(4k, 2k-1)$-colorable. Then G does not have a homomorphism to the g'-cycle for $g' \geq 2\lceil\frac{2k-1}{2}\rceil + 1 = 2k+1$. Assume k is odd. Then $g \geq 3k-1$ and $g' = 2k+1$, and G does not have a homomomorphism to the g'-cycle for $g' > \frac{2g+2}{3}$.

The graphs in [1] are not subcubic. They are constructed as a series of graphs $I_j^{(i)}$, where $0 \leq j \leq c_1 g$ and $0 \leq i \leq \log_2 g$. The graphs for $i = 0$ are paths of lengths at least $c_2 g$. The graphs for $i+1$ are obtained from graphs for $i' \leq i$ by at most c_3 series-parallel combinations.

Assume $c_2 g \geq 2c_3 \log_2 g$. When combining two $I_j^{i'}$ in series, we proceed as in [1]. When combining them in parallel, however, we modify the proof and identify the two endpaths of lengths $\ell_i = c_3((\log_2 g) - i)$. Clearly this modified G' and is an image of G, so G' is not mappable to the g'-cycle either. The c_3 combinations in step i are done at distances $\ell_i, \ell_i - 1, \ldots, \ell_i - c_3 + 1$, then we proced to step $i+1$ starting at distance $\ell_{i+1} = \ell - c_3$. Clearly G' is subcubic and does not have a homomorphism to the g'-cycle. The girth of G' is at least $\hat{g} \geq g - 4c_3 \log_2 g$ to account for combining two paths and at both ends, or $g \leq \hat{g} + c\log\hat{g}$. Then G does not have a homorphism to the g'-cycle for $g' > \frac{2\hat{g}+2+2c\log\hat{g}}{3}$. □

A simple application of Theorem 10 is then the following one.

Corollary 8 *There exists a constant c such that for all $g \geq 11$ there exists a subcubic series-parallel graph G of girth g such that G does not have a (k,d)-coloring for all k, d satisfying $\frac{k}{d} < 2 + \frac{3}{g-c\log g}$.*

References

[1] C. Chien and X. Zhu, The circular chromatic number of series-parallel graphs with large girth, J. Graph Theory 33 (2000), 185–198.

[2] Z. Dvořák, J. Sereni and J. Volec, Subcubic triangle-free graphs have fractional chromatic number at most 14/5, J. Lond. Math. Soc. 89 (2014) 641–662.

[3] S. Fajtlowicz, On the size of independent sets in graphs, Congr. Numer. 21 (1978), 269–274.

[4] C.C. Heckman and R. Thomas, Independent sets in triangle-free cubic planar graphs, J. Combin. Theory Ser. B 96 (2006), 253–275.

[5] P. Hell and X. Zhu, The circular chromatic number of series-parallel graphs, J. Graph Theory 33 (2000), 14–24.

[6] Z. Pan and X. Zhu, The circular chromatic number of series-parallel graphs of large odd girth, Discrete Math. 245 (2002), 235–246.

[7] X. Zhu, Bipartite subgraphs of triangle-free subcubic graphs, J. Combin. Theory Ser. B 99 (2009), 62–83.