Graceful Isometric Embeddings of Trees on n-cubes

Tomás Feder[*] Carlos Subi[†]

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It has been conjectured that for every tree T with n edges and $n+1$ vertices, there exists a labeling of the vertices of T with label x_i assigned to v_i in such a way that $x_i \in \{0, 1, 2, \ldots, n\}$ with $x_i \neq x_j$ for $v_i \neq v_j$, and if each edge v_iv_j of T is assigned label $|x_i - x_j|$ then the labels of the n edges are the distinct integers $1, 2, \ldots, n$ up to permutation. Such a labeling is called *graceful*, and the conjecture is called the *graceful tree conjecture*. This conjecture has been verified for trees that are caterpillars.

In this note we connect graceful labellings with embeddings in the n-cube. The *n-cube* is the graph with 2^n vertices v_i for $i = 0, 1, 2, \ldots, 2^n - 1$, where i can be represented in binary as $x_1^i x_2^i \cdots x_n^i$, and $n2^{n-1}$ edges v_iv_j if the binary representations of i and j differ in exactly one bit d, i.e., $x_b^i \neq x_b^j$ if and only if $b = d$ for $1 \leq b \leq n$, or equivalently, $|i - j| = 2^d$ for some $1 \leq d \leq n$.

A *labeling* of the n-cube is an assignment of distinct values a_i to the vertices v_i and of values $b_{ij} = |a_i - a_j|$ to the edges v_iv_j of the n-cube. The *canonical labeling* of the n-cube has $a_i = i$. A *graceful embedding* of a graph G in a labeled n-cube is an isomorphism from G to a subgraph G' of the n-cube such that all edges of G' have distinct labels in the n-cube.

A graph G has a graceful embedding in the canonically labeled n-cube if and only if G is a tree with at most n edges, and in this case the tree G' is an isometric subgraph of the n-cube, that is, every edge of G' flips a different bit d in the binary representation of the n-cube.

To see this, note that if G has a cycle, then the corresponding cycle of G' flips the same bit d at least twice, so the label 2^d repeats in the edges of G'. Thus G is a tree, and as there are only n distinct labels of edges, namely $1, 2, 4, \ldots, 2^{n-1}$, the tree G must have at most n edges. If we embed such a tree G with at most n edges by flipping a different bit d for each edge starting with an assigned vertex of G' for a chosen root, we obtain an isometric subtree G' with each of the labels 2^d occurring exactly once.

We conjecture that given any labeling of the n-cube, every tree G with at most n edges has a graceful embedding G' in the n-cube such that the tree G' is an isometric subgraph of the n-cube. In brief, we say that such a tree has a *graceful isometric embedding* in the n-cube.

[*]268 Waverley St., Palo Alto, CA 94301. E-mail: tomas@theory.stanford.edu.

[†]E-mail: carlos subi@hotmail.com.

We show a special case of the conjecture. Label the vertices of the tree G as $u_0, u_1, \ldots, u_k$ in such a way that $u_0, u_1, \ldots, u_i$ induces a subtree T_i of G for $1 \leq i \leq k$. Suppose that for each $1 \leq i \leq k$, the leaf u_i of T_i is adjacent to a vertex u_j that has degree d in T_i, and furthermore $3i - d - 2 < n$ if $j \neq 0$ and $2i - d - 1 < n$ if $j = 0$. Then G has a graceful isometric embedding in the n-cube.

To see this, embed u_0 as v_0. Once T_{i-1} has been embedded, using $i - 1$ distinct labels, note that the labels occur at most twice in the n edges coming out of u_j, so for the $d - 1$ edges coming out of v_j in T_i we must avoid the two edges labeled by each of these $d - 1$ labels to go from u_j to u_i, we must also avoid the two edges labeled by the remaining $i - d$ labels, and finally also avoid the bit position of these $i - d$ labels. The requirement is thus $2(d - 1) + 3(i - d) < n$, that is, $3i - d - 2 < n$. If $j = 0$ and u_j is embedded as v_0, then the labels out of u_j are all distinct, so we must only avoid the edge labelled by each of the $d - 1$ labels, the edges labeled by each of the remaining $i - d$ labels, and finally also aviod the bit position of these $i - d$ labels. The requirement in this case is thus $d - 1 + 2(i - d) < n$, that is, $2i - d - 1 < n$. This completes the proof.

A special case of this result is the n-edge star G, since the center of the star can be chosen as u_0, and then $d = i$, giving the requirement $2i - d - 1 < n$, that is $2i - i - 1 = i - 1 < n$ for $1 \leq i \leq n$, as required.

A second special case of this result is the case where the tree G has $k \leq 2 + n/3$ edges for $n \geq 4$, since we may choose u_0 adjacent to a leaf chosen as u_k, so for $i = 1$ the result is obvious, for $2 \leq i < k$ we have $d \geq 2$ and $i \leq k - 1$, the requirement is $3(k - 1) - 2 - 2 < n$, or $k < (n + 7)/3$, or $k \leq (n + 6)/3$, and for $i = k$ we have $d \geq 2$, and the requirement is $2k - 2 - 1 < n$, or $k < (n + 3)/2$, or $k \leq (n + 2)/2$, which follows since $k \leq 2 + n/3$ imples $k \leq 1 + n/2$ for $n \geq 4$.

A third case of the conjecture that can be derived is the case where $n \leq 3$ and G is an arbitrary tree. As stars were already derived, only the case of a path with 3 edges remains to be shown, with $n = 3$. We may assume the least label is 0, and the labels of the neighbors of 0 are $0 < a < b < c$. If the label of the neighbor of a, b other than 0 is x, we may consider the path $c, 0, a, x$ unless $x = 2a$ or $x = a + c$, or the path $c, 0, b, x$ unless $x = 2b$ or $x = b + c$, leaving only the case $x = a + c = 2b$. If the label of the neighbor of b, c other than 0 is y, we may consider the path $a, 0, b, y$ unless $y = b - a$ or $y = b + a$, or the path $a, 0, c, y$ unless $y = c - a$ or $y = c + a$ or $y = 2c$, leaving only the case $y = b + a = c - a$. Solving $a + c = 2b$ and $c = 2a + b$ gives $b = 3a$ and $c = 5a$, and so $x = 6a$ and $y = 4a$. If the neighbor of x, y other than b is z, we have the path $0, a, x, z$, or $0, a, 6a, z$, which works unless $z = 7a$ or $z = 11a$, in which cases the path $0, c, y, z$, or $0, 5a, 4a, z$ works. This completes the proof.

It is not possible to always choose this path of length 3 to have 0 in the second position, since in the proof above we may choose the neighor of a, c other than 0 to be $t = 2a$, and setting $a = 1$ gives the case $a = 1, b = 3, c = 5, t = 2, y = 4, x = 6, z = 7$.

It is also not possible to always choose this path of length 3 starting at 0. If the neighbors of 0 are $1, 2, 3$, the neighbors of $1, 2$ are $0, 4$, the neighbors of $1, 3$ are $0, 6$, the neighbors of $2, 3$ are $0, 5$, and the neighbor of $4, 5, 6$ is 7, then none of the paths of length 3 from 0 to 7 works.

We conclude by observing that the canonical labeling of the n-cube decomposes the edges of the n-cube into n perfect matchings, one for each bit position. Consider any decomposition of the edges of the n-cube into n perfect matchings. One may ask whether any tree G with at most n edges can be embedded isometrically into the n-cube so that each edge of G' belongs to a different perfect matching out of the n chosen perfect matchings.

The answer is again positive if $n \leq 3$ by inspection, as one can consider the only decomposition of the 3-cube into perfect matchings that is not the decomposition by bit positions.

The answer is also positive if G is a star, as the edges incident to the vertex chosen for the center all belong to perfect matchings.

The answer is again positive if the tree G has at most $\lceil n/2 \rceil$ edges, since in order to connect edge $i \leq \lceil n/2 \rceil$ we may consider the $n-i+1$ bit positions not yet used, and the $n-i+1$ matchings not yet used, and these two will intersect, as $2(n-i+1)-n = n-2(i-1) \geq 1$.

The answer is however negative if $n = 4$ and G is a tree with three paths coming out of the root, one of them of length 2 and the other two of length 1. Suppose we have embedded the star with three leaves and wish to add the extra edge to a given leaf x to complete the embedding of G. Then the edge coming out of x must belong to the same matching and same dimension as the edge coming out of the root of the star that was not used at the root. In other words, we must have a square with opposite edges in the same matching. However, there exists a choice of the four matchings in the 4-cube that has no such square, namely for each dimension, the 8 edges across the dimension are decomposed into two antipodal edges (opposite in the cube) for each of the four matchings.

More generally, the answer is still negative if $n = 4k, k \geq 1$, since in these cases we can still avoid having squares with opposite edges in the same matching by grouping the dimensions into groups of four, and using four matchings for each group of four dimensions. Thus each group of four four matchings forms a collection of 4-cubes, with each 4-cubes decomposed as in the case for $n = 4$, and 4-cubes with an odd number of 1s in the other $n-4$ dimensions given a different labelling from 4-cubes with an even number of 1s in the other $n-4$ dimensions, obtained by permuting the four matchings, so no square with opposite edges in the same matching occurs.

Once this construction of the matching decomposition for $n = 4k, k \geq 1$ is done, the counter example is again the tree G consisting of $n-1$ paths coming out of the root, one of length two and the other $n-2$ of length one, so the edge at the end of the path of length two would have to belong to a square with the opposite in the square in the same matching.

We have observed that for $n = 4$ and any decomposition into four perfect matchings, the isometric path G of length four always works, so we may conjecture that for any n the isometric path of length n always works, which is left open.