Graceful Almost Symmetrical Trees

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1 Introduction

Suppose a graph G has $n+1$ vertices and n edges. Suppose the vertices of G are labelled $0, 1, 2, \ldots, n$. We say that such a labelling of G is a *graceful labelling* if when we assign to each edge $e = (u, v)$ of G the label $|u - v|$ (where we identify u, v with their labels u, v) we have that the n edges of G get precisely the distinct labels $1, 2, \ldots, n$. We say that G is *graceful* if G has a graceful labelling. The following has been conjectured.

Conjecture 1 *(Graceful Tree Conjecture) Every tree is graceful.*

The graceful tree conjecture has been verified for trees that are *caterpillars* and in many other cases, see [1] for a survey. In this note we prove the conjecture for a class of trees that we call *almost symmetrical*, thus generalizing results for symmetrical trees from [2, 3].

2 Almost Symmetrical Trees

A caterpillar is a tree consisting of a path $a_1, \ldots, a_m$ plus an arbitrary number of leaves attached to the vertices a_i.

Suppose T is a graceful tree that has $n+1$ vertices, with p vertices in one side A of the bipartition and $n+1-p$ vertices in the other side B of the bipartition. We say that a graceful labelling of T is *bipartite* if the vertices in A have labels $0, 1, \ldots, p-1$ and the vertices in B have labels $p, p+1, \ldots, n$.

A (k, r) symmetrical star consists of a root and k paths of length r attached at the root, for a total of $kr + 1$ vertices. We note that the $(3, 2)$ symmetrical star does not have a bipartite graceful labelling. It is known that caterpillars T have a bipartite graceful labelling with the label 0 at a_1.

A graceful labelling of a tree T is *nearly bipartite* if T has $2r+1$ vertices for some $r \geq 0$, labelled $0, 1, \ldots, 2r$, the vertex 0 has degree two and is joined precisely to vertices r and $2r$, and all remaining edges join vertices among $1, \ldots, r$ to vertices among $2, \ldots, r$.

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Theorem 1 *Suppose trees $T_0, T_1, \ldots, T_k$ are gracefully labelled, tree T_0 has $r+1$ vertices with 0 at a leaf, and trees T_i for $1 \le i \le k$ have $2r+1$ vertices and a nearly bipartite graceful labelling. Then the tree T obtained by joining the T_i at 0 has a graceful labelling with the common vertex of the joined T_i having label 0 for T. Also the tree T' obtained by joining the T_i with $i \ge 1$ at 0 has a graceful labelling with the common vertex of the joined T_i having label 0 for T'.*

PROOF: Relabel the vertices of T_0 other than 0 by replacing label x with $x + kr$. Relabel the vertices of T_i with labels y among $1, \ldots, r$ by $y + (i-1)r$. Relabel the vertices of T_i with labels z among $r+1, \ldots, 2r$ by $z + (2k-i)r$.

The resulting labels will be precisely $0, \ldots, (2k+1)r$, and the edge labels are as follows. The edge labels that are multiples of r join 0 and occur $r, (2k+1)r$ in T_1, $2r, 2kr$ in T_2, ..., $kr, (k+2)r$ in T_k, and $(k+1)r$ in T_0. The edge labels w that are not multiples of r occur, excluding the vertex 0, from 1 to $r-1$ in T_0, from $r+1$ to $3r-1$ without $2r$ in T_k, from $3r+1$ to $5r-1$ without $4r$ in T_{k-1}, $ldots$, from $(2k-1)r+1$ to $(2k+1)r-1$ without $2kr$ in T_1.

This completes the case of T. For T' without T_0, the construction is essentially the same, we relabel the vertices of T_i with labels z among $r+1, \ldots, 2r$ instead by $z + (2k-1-i)r$ and the proof is similar. ■

Call a vertex of a gracefully labelled T *even* if it is at even distance from the vertex labelled 0, and *odd* if it is at odd distance from the vertex labelled 0. Call two gracefully labelled trees T_1, T_2 with the same number of vertices *essentially the same* if (1) each label x is either in an even vertex in both T_1, T_2, or in an odd vertex in both T_1, T_2; and (2) if (x, y) is the edge of label v in T_1 and (x', y') is the edge of label v in T_2, with the labels satisfying $x < y$ and $x' < y'$, then x, x' are either both even vertices or both odd vertices.

Theorem 2 *If T_1 and T_2 are labelled essentially the same, and 0 is a leaf for both T_1 and T_2, then the tree T obtained by joining T_1 and T_2 at 0 has a nearly good graceful labelling with 0 at the joint vertex.*

PROOF: Suppose T_1 and T_2 have $r+1$ vertices labelled $0, \ldots, r$. Relabel all even vertices of T_1 other than 0 by replacing label x with $x+r$. Relabel all odd vertices of T_2 other than 0 by replacing label y with $y+r$. Then if edges (x, y) in T_1 and (x', y') in T_2 have the same label $v < r$ with $x < y, x' < y'$, and say x, x' are even other than 0, they become $(x+r, y)$ and $(x', y'+r)$, with labels $r - v$ and $r + v$ respectively. This covers all edge labels other than $r, 2r$, while $(0, r)$ is in T_1 and $(0, 2r)$ is in T_2 after relabelling for labels $r, 2r$. ■

Corollary 1 *Suppose $T_0, \ldots, T_k$ are gracefully labelled trees with $r+1$ vertices and 0 at a leaf, and T_i, T_{k-i} are essentially the same for all $0 \le i \le k$. Then T obtained by joining all T_i at 0 has a graceful labelling with 0 at the joint common vertex.*

PROOF: First apply Theorem 2 to join T_i, T_{k-i} for all i and obtain $\lceil k/2 \rceil$ nearly good gracefully labelled trees with $2r+1$ vertices, plus $T_{k/2}$ with $r+1$ vertices if k is even. These can then be joined by Theorem 1 ■

A tree T rooted at v is *symmetrical* if all vertices of T in the same level have the same degree. It was shown by Koh, Rogers and Tan [2] and by Poljak and Sura [3] that symmetrical trees are graceful. Call a tree T with root v *almost symmetrical* if each non-leaf vertex x with $k_x + 1$ children rooting subtrees $T_0, \dots, T_{k_x}$ is such that T_i and $T_{k_x - i}$ are isomorphic rooted trees, and all T_i have the same number of vertices, for $0 \leq i \leq k_x$.

Corollary 2 *Almost symmetrical trees have a graceful labelling with the root having label* 0.

PROOF: Suppose inductively that for a vertex x the subtrees $T_0, \dots, T_{k_x}$ are labelled with root v_i having label 0 and have r vertices labelled $0, \dots, r-1$. Obtain for each T_i the tree T'_i with new root w_i whose only child is v_i, and give to w_i label r. Replace in each T'_i each label x with $r - x$, so the root w_i has label 0, and join the T'_i at the roots w_i to obtain the subtree rooted at x and its labelling with x assigned label 0 by Corollary 1. ■

Corollary 3 *Suppose trees $T_0, T_1, \dots, T_k$ are gracefully labelled, tree T_0 has $r+1$ vertices with 0 at a leaf, and trees T_i for $1 \leq i \leq k$ have $2r+1$ vertices and a nearly bipartite graceful labelling. Then the tree T obtained by joining considering an edge e and joining some the T_i at 0 at one endpoint of e and the remaining T_i at 0 and at the other endpoint of e has a graceful labelling with the edge e having labels 0 and $(2k+1)r+1$. A similar result follows for the T' obtained similarly from an edge e' with T_0 omitted and the edge e' having labels 0 and $2kr+1$.*

PROOF: We proceed as in Theorem 1 to join at 0 and add the edge $e = (0, (2k+1)r+1)$. We then switch T_i from root 0 to root $(2k+1)r+1$ by replacing label x with label $(2k+1)r+1-x$ in such trees T_i. A similar proof holds if T_0 is omitted for $e = (0, 2kr+1)$. ■

A tree with $n+1$ vertices is *strongly gracefully labelled* if it has a perfect matching and the endpoints of each edge e of the matching are labelled x and $n-x$ for some x depending on e.

Corollary 4 *Suppose $T_0, \dots, T_k$ are gracefully labelled trees with $r+1$ vertices and 0 at a leaf, with the labelling of T_i obtained from a strongly graceful labelling of $T_i - \{0\}$ by replacing each x with $r-x$ and attaching 0 to r. Suppose also that T_i, T_{k-i} are the same tree for all $0 \leq i \leq k$. Then T obtained by choosing an edge e and attaching all T_i at 0 at an endpoint of e has a strongly graceful labelling with labels $0, kr+1$ at the endpoints of e.*

PROOF: The proof is as in Corollary 3. ■

Corollary 5 *Let S be a (k,r) symmetrical star, let S' be a (k',r) symmetrical star, and let T be obtained by joining the two roots of S and S' with an edge. If at least one of k, k', r is even then T is graceful with the two roots assigned 0 and $r(k+k')+1$.*

PROOF: Follows from Corollary 3 if one of k, k' is even and from Corollary 4 if r is even. ■

References

[1] M. Edwards and L. Howard, A survey of graceful trees, Atlantic Electronic Journal of Mathematics, Volume 1, Number 1, Summer 2006, 5–30.

[2] K.M. Koh, D.G. Rogers, and T. Tan, On graceful trees, Nanta Math. 10 no. 2 (1977), 207–211.

[3] S. Poljak and M. Sura, An algorithm for graceful labelling of a class of symmetrical trees, Ars Combin. 14 (1982), 57–66.