# Graceful Labelings of Paths on Grids

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## 1 Introduction

Graceful labelings of trees have been extensively studied [1]. We generalize this notion as follows. Let $K$ be a clique on $n$ vertices whose edges have been labeled with $k$ distinct colors. We say that a graph $G$ with $n$ vertices and $k$ edges is $K$-graceful if $G$ can be embedded in $K$ by a one-to-one correspondence from $V(G)$ to $V(K)$ in such a way that the $k$ edges of $G$ get assigned the $k$ distinct colors in the corresponding edges of $K$.

We show that if $k = n - 1$ and $G$ is a path with $n$ vertices and $k = n - 1$ edges, then whether $G$ is $K$-graceful is NP-complete. By contrast, if $G$ is an $n$-vertex star or differs from such a star by substitution of a number of edges bounded by a constant, then the problem is polynomial time solvable.

We then consider the *grid* $K$ *graph* defined as follows. The vertices of $K$ are tuples $(i_1, \ldots, i_k)$, where $0 \leq i_j < n_j$ and the positive integers $k$ and $n_j$ are given, for $1 \leq j \leq k$. The colors are the tuples $(a_1, \ldots, a_k)$ for $0 \leq a_i < n_j$ other than the all-zero tuple, and an edge $(i_1, \ldots, i_k)(i'_1, \ldots, i'_k)$ is assigned color $(|i_1 - i'_1|, \ldots, |i_k - i'_k|)$.

The standard graceful labeling problem has $k = 1$, so we are labeling by a grid that is just a path, and it is conjectured that every tree can be labeled on such a path with the same number of vertices $n_1$.

Here we show that stars can always be gracefully labeled on any such grid $K$ graph. On the other hand, we show that paths can not be gracefully labeled on the $k$-cube $K$ with $k \geq 2$, where the $k$-cube $K$ is the grid $K$ with all $n_j = 2$ for $1 \leq j \leq k$. We conjecture that paths can always be gracefully labelled on any grid $K$ that is not such a cube. We have verified this for the cases with $k = 2$ and either $n_1 = 2, 3 \leq n_2 \leq 11$, or $n_1 = 3, n_2 = 3$. For $k = 2$ and $n_1 = 2, n_2 \geq 3$ we could always decompose the grid into two paths that use different colors, but have no general technique for joining the ends of two paths.

## 2 NP-completeness

**Theorem 1** *The question of whether a path is $K$-graceful is NP-complete.*

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PROOF: Consider the question of whether a bipartite graph $G = (U, V, E)$ with $|U| = p$ and $|V| = p + 1$ has a Hamiltonian path is NP-complete [2]. Define $K$ with $n = 2p + 2(p + 1) + 1 = 4p + 3$ vertices and $k = 3p + (p + 1) + 1 = 4p + 2 = n - 1$ colors by assigning to each vertex $v \in U \cup V$ two vertices $x_u, y_u$ in $K$, plus an additional vertex $z$, assigning to each edge $x_u y_u$ a different color $a_u$, and if there are edges $uv \in E$ with $u \in U$ and $v \in V$, choose colors $b_u, c_u$, assign to $x_u x_v$ and $y_u y_v$ color $b_u$, and to $x_u y_v$ and $y_u x_v$ color $c_u$. Finally assign to all remaining edges of $K$ a new color $d$.

If $G$ has a Hamiltonian path $v_1 u_1 \cdots v_p u_p v_{p+1}$, then the resulting path for $K$ starts at $z$ and visits then the two corresponding vertices $x_{v_i}, y_{v_i}$ or $x_{u_i}, y_{u_i}$ in the order of the path, entering for $u_i$ via an edge of color $b_{u_i}$, then using color $a_{u_i}$, and exiting via color $c_{u_i}$ to go to color $a_{v_{i+1}}$ for the two vertices corresponding to $v_{i+1}$. This uses each of the $k$ colors exactly once.

Conversely, color $d$ is needed to visit $z$ so $z$ must be at the end of the path for $K$, and then we must visit the edges for $u_i, v_i$ of colors $a_{u_i}, a_{v_i}$ in turn, giving a Hamiltonian path for $G$. ■

**Theorem 2** *If $G$ differs from a star in a constant number of edges, then whether $G$ is $K$-graceful is polynomial time solvable.*

PROOF: If $G$ is a star, then we just need to determine whether $k = n - 1$ distinct colors occur for at least one of the $n$ distinct vertices of $K$. If $G$ differs from a star by $c$ edges, then try a star at each of the $n$ distinct vertices of $K$ and do the $c$ edge switches in $n^{O(c)}$ possible ways to test. ■

# 3 Paths on Grids

**Theorem 3** *A star is grid $K$-graceful.*

PROOF: Just put the center of the star at $(0, 0, \ldots, 0)$. ■

**Theorem 4** *A path is not $k$-cube $K$-graceful for $k \geq 2$.*

PROOF: The $2^k - 1$ vectors corresponding to the $k$ colors add up to $2^{k-1} = 0$ modulo 2, so the paths would end up in the starting vertex of the path, contrary to each vertex of the $k$-cube being visited once. ■

**Conjecture 1** *Paths are grid $K$ graceful if $K$ is not the $k$-cube for $k \geq 2$.*

We have verified the conjecture for $k = 2, n_1 = 2, 3 \leq n_2 \leq 11$. For example, if $n_2 = 3$, we can use the path $(0, 0), (0, 2), (1, 0), (1, 1), (0, 1), (1, 2)$. For $n_2 \geq 3$, we have so been able to use the edges $(0, a)$ with $a$ even starting at $(0, 0)$ with the largest even $a$ and successively decreasing $a$ even in alternating directions, and similarly the edges $(0, b)$ with $b$ odd starting at $(1, 0)$ with the largest odd $b$ and successively decreasing $b$ odd in alternating directions. If we try a similarly alternating scheme for $(1, a)$ with $a$ even and $(1, b)$ with $b$ odd to join the remaining vertices, we end up with four paths that can be merged into two paths without using the $(1, 0)$ edge, but this edge cannot be appropriately placed. Thus the pattern for $(1, a)$ and $(1, b)$ edges seems rather arbitrary.

We have also verified the conjecture for $k = 2, n_1 = 3, n_2 = 3$ by giving the path $(0, 0), (2, 2), (2, 0), (1, 1), (1, 0), (0, 2), (2, 1), (0, 1)$.

# References


[1] M. Edwards and L. Howard, A survey of graceful trees, Atlantic Electronic Journal of Mathematics, Volume 1, Number 1, Summer 2006, 5–30.

[2] M.R. Garey and D.S. Johnson, Computers and Intractability, A Guide to the Theory of NP-Completeness, Freeman (1979), 340 pp.