Nearly Tight Bounds on the Number of Hamiltonian Circuits of the Hypercube and Generalizations

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Abstract

It has been shown that for every perfect matching M of the d-dimensional n-vertex hypercube, $d \geq 2, n = 2^d$, there exists a second perfect matching M' such that the union of M and M' forms a Hamiltonian circuit of the d-dimensional hypercube. We prove a generalization of a special case of this result when there are two dimensions that do not get used by M. It is known that the number M_d of perfect matchings of the d-dimensional hypercube satisfies $M_d = (\frac{d}{e}(1 + o(1)))^{n/2}$ and, in particular, $(2d/n)^{n/2}(n/2)! \leq M_d \leq (d!)^{n/(2d)}$. It has also been shown that the number H_d of Hamiltonian circuits of the hypercube satisfies $1 \leq \lim_{d\to\infty}(\log H_d)/(\log M_d) \leq 2$. We finally strengthen this result to a nearly tight bound $((d \log 2/(e \log\log d))(1 - o(1)))^n \leq H_d \leq (d!)^{n/(2d)}((d-1)!)^{n/(2(d-1))}/2$ proving that $\lim_{d\to\infty}(\log H_d)/(\log M_d) = 2$. This means that the bound $H_d \geq M_d$ is improved to a nearly tight $H_d = M_d^{2-o(1)}$, so the number of Hamiltonian circuits in the hypercube is nearly quadratic in the number of perfect matchings. The proofs are based on a result for graphs that are the Cartesian product of squares and arbitrary bipartite regular graphs that have a Hamiltonian cycle. Don Knuth [12] recently improved the lower bound for the number of Hamiltonian circuits in the hypercube to $H_d \geq ((d/(4e)) - O(\log^2 d))^n$. We also study a labeling scheme related to matchings.

1 Introduction

We study properties of matchings and Hamiltonian cycles in various classes of graphs, including Cartesian products of graphs that generalize the hypercube and regular bipartite and non-bipartite graphs.

A perfect matching M in a graph G is a collection of edges in G such that each vertex of G is incident to precisely one edge of M. A Hamiltonian cycle H in a graph G is a collection of edges in G that induce a connected subgraph of G such that each vertex of G is incident to precisely two edges of H.

Given two graphs G and H, the Cartesian product $G\Box H$ has $V(G\Box H) = V(G) \times V(H)$ and $E(G\Box H) = \{(zx_H, zy_H) : (x_H, y_H) \in E(H)\} \cup \{(x_G t, y_G t) : (x_G, y_G) \in E(G)\}$. Let K_2 be the complete bipartite graph on two vertices $0, 1$. The d-dimensional hypercube is the Cartesian product of d copies of K_2 and has $n = 2^d$ vertices $x = x_1 \cdots x_d$ with $x_i \in \{0, 1\}$ for $1 \leq i \leq d$, where two vertices are adjacent if they differ in precisely one x_i.

For a balanced bipartite graph $G = (U, V, E)$ where $|U| = |V| = n$, the bipartite adjacency matrix $A = A(G) = [a_{uv}]$ is the $n \times n$ matrix with $a_{uv} = 1$ if $uv \in E$ and $a_{uv} = 0$ if $uv \notin E$ for $u \in U, v \in V$.

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Independently, Fisher[10] and Kastelyn[11] proved that the number of perfect matchings of G is the permanent of $A(G)$ when G is a balanced bipartite graph with adjacency matrix $A(G)$. Brègman[1] proved the conjecture of Minc[14] that for any $n \times n$ $0,1$-matrix A with row sums $r_1, \ldots, r_n$, the permanent of A is at most $\prod_{i=1}^{n} (r_i!)^{1/r_i}$. In particular, a d-regular bipartite n-vertex graph has at most $(d!)^{n/(2d)} = (\frac{d}{e}(1+o(1)))^{n/2}$ $(*)$ perfect matchings.

Independently, Egoryčev [6] and Falikman [7] proved the conjecture of van der Waerden [17] that for any doubly stochastic $n \times n$ matrix A the permanent of A is at least $n!/n^n$. This was used by Clark, George and Porter [3] to show that the number of perfect matchings of a d-regular bipartite n-vertex graph is at least $(2d/n)^{n/2}(n/2)! = (\frac{d}{e}(1+o(1)))^{n/2}$ $(**)$.

It was conjectured by Kreweras [13] that for any perfect matching M of the d-dimensional hypercube, $d \geq 2$, there exists a second perfect matching M' such that the union $H = M \cup M'$ forms a Hamiltonian cycle. This was shown by Fink [9]. Let M_d be the number of perfect matchings of the hypercube and let H_d be the number of Hamiltonian cycles of the hypercube. It was shown by Perezhogin and Potapov [16] that $(n/2)((\log d) - 1 + o(1)) \leq \log H_d \leq n((\log d) - 1 + o(1))$, so that $(\log H_d)/(\log M_d)$ tends assymtotically to values between 1 and 2. We shall show in this paper that the limit is actually 2, ending a series of earlier results [2, 4, 5, 15]. We thus prove that $\lim_{d\to\infty}(\log H_d)/(\log M_d) = 2$. This means that the bound $H_d \geq M_d$ is improved to a nearly tight $H_d = M_d^{2-o(1)}$. Thus the number of Hamiltonian circuits in the hypercube is nearly quadratic in the number of perfect matchings.

2 Balanced Labeling Matching Partitions

The d-dimensional hypercube can be decomposed into d perfect matchings, one for each dimension. If we orient each matching from 0 to 1, then the orientation of the edges of each matching gives a label 0 or 1 to each dimension and thus a univocous label $x = x_1 \cdots x_d$ with $x_i \in \{0, 1\}$ for $1 \leq i \leq d$ for the vertices. It is natural to ask whether such a univocous label can similarly be obtained from any decomposition of the hypercube into perfect matchings. We shall answer this question and a generalization to regular bipartite graphs affirmatively.

Every d-regular bipartite graph is the union of d edge-disjoint perfect matchings. Suppose more generally that G is an n-vertex graph that is the union of d edge-disjoint perfect matchings $M_1, M_2, \ldots, M_d$. A *labeling orientation of* G is an assignment of directions to the edges of G and a corresponding assignment of labels $x = x_1 x_2 \cdots x_d$ to each vertex v of G, so that if the edge e of M_i incident to v is outgoing then $x_i = 0$ and if e is incoming then $x_i = 1$. A *balanced labeling orientation of* G is a labeling orientation of G such that the number of vertices having any given label x is either $\lfloor n/2^d \rfloor$ or $\lceil n/2^d \rceil$. Notice that if the d-dimensional hypercube, with $n = 2^d$ vertices, is decomposed into d perfect matchings corresponding to the d dimensions, and dimension i is oriented from 0 to 1 in the ith bit position, then we obtain a balanced labeling orientation of the hypercube that assigns to each vertex v its corresponding coordinates $x = x_1 x_2 \cdots x_d$.

Theorem 1 *Suppose that G is an n-vertex graph that is the union of d edge-disjoint perfect matchings $M_1, M_2, \ldots, M_d$. Then G has a balanced labeling orientation.*

Proof. Orient the edges of M_1 arbitrarily. This assigns to x_1 the value 0 to $n/2$ of the vertices and 1 to the other $n/2$ vertices. Suppose inductively that we have already oriented the first t matchings M_i, $1 \leq i \leq t$ in a balanced manner, so that each label occurs $\lfloor n/2^t \rfloor$ or $\lceil n/2^t \rceil$ times. We greedily select edges of M_{t+1} and orient them as follows. Select an edge $e = uv$ of M_{t+1} and orient it arbitrarily, say from u to v. Suppose u and v have labels x and y respectively for the first t bits of the label. If $x = y$, then $x' = xx_{t+1} = x0$ and $y' = xy_{t+1} = x1$, and we have made progress towards splitting the label x evenly. If $x \neq y$, then select another edge $e' = u'v'$ of M_{t+1} such that the label of u' is also y, if such an e' exists, and orient e' from u' to v', so in this case we have made progress towards splitting y evenly. If no such e' exists, then the number k of vertices with label y was odd and we have split k as $\lfloor k/2 \rfloor$ and $\lceil k/2 \rceil$ for labels $y0$ and $y1$ respectively, as required. If e' exists, then either v' has the same label x as u in which case we have again made progress towards splitting x evenly as $x0$ and $x1$, or v' has label $z \neq x$ and we proceed inductively to look for an edge $e'' = u''v''$ with u'' having the same label z has v'. Eventually the process ends in some edge $e^i = u^i v^i$. If the label of v^i is x then we have made progress towards splitting x evenly as before, otherwise the number of vertices with the label of v^i was odd and split evenly as floor and ceiling as before. In this last case the only imbalance is at u with label $x0$, so we start looking for $e' = u'v'$ where u' has label x, and orient e' from v' to u', making progress towards splitting x evenly. We proceed with the imbalance at v' with label $y0$ as we just did for x to make progress towards splitting y similarly. In the end, each label x with k vertices will have been split into two labels $x0$ and $x1$ having one $\lfloor k/2 \rfloor$ vertices and the other one $\lceil k/2 \rceil$ vertices, completing the induction. □

Corollary 1 *If the d-dimensional hypercube with $n = 2^d$ vertices is decomposed into d edge-disjoint perfect matchings, then a balanced labeling orientation exists and assigns each label $x = x_1 x_2 \cdots x_d$ exactly once.*

Proof. The result follows from Theorem 1 and the fact that $\lfloor n/2^d \rfloor = \lceil n/2^d \rceil = 1$. □

3 Hamiltonian Circuits and Isomorphisms

The following applies to the whole section. Let $G = (U, V, E)$ be a bipartite graph with $|U| = |V| = r = 2k$ having a Hamiltonian circuit C. We may label the vertices of each of U, V as $1, 2, \ldots, r$. We may also decompose C as the union of two perfect matchings M and M', and view M and M' as two permutations p and p' on $1, 2, \ldots, r$, so that $p(i) = j$ and $p'(i') = j'$ if vertex i in U is matched to j in V by M, and vertex i' in U is matched to j' in V by M'.

Theorem 2 *Of the two permutations p and p', one is odd and the other one even. Thus the graph joining pairs of matchings M and M' if they jointly form a Hamiltonian circuit is bipartite, and if M_G and H_G are the numbers of perfect matchings and Hamiltonian circuits of G, respectively, then $H_G \leq M_G^2/4$.*

Proof. We may relabel the vertices so that p and p' become q and q' with $q(i) = i$ and $q'(i) = i+1$ modulo r. Thus there exist permutations s and t such that $p = sqt = st$ and $p' = sq't$, where s and t permute vertices of U and V respectively. An even permutation is a product of an even number of transpositions, and an odd permutation is a product of an odd number of transpositions. The proof is completed by observing that $q' = (12 \cdots r) = (12)(13) \cdots (1r)$ is odd since $r - 1 = 2k - 1$.

Thus the two perfect matchings M and M' forming a Hamiltonian cycle are chosen from M_1 and M_2 possible matchings such that $M_1 + M_2 = M_G$ respectively, so $H_G \leq M_1 M_2 \leq M_G^2/4$. □

This improves by a factor of two the bound $M_G^2/2$ of Clark [2] on the number of Hamiltonian circuits, which is also an improvement on earlier results by Dixon and Goodman [4], Douglas [5], and Mollard [15]. We improve this bound further.

Theorem 3 *Let G be a d-regular bipartite graph $G = (U, V, E)$ with $|U| = |V| = n/2$. Then for $k \leq d$, the number of sequences $M_1, \ldots, M_k$ of k edge-disjoint perfect matchings for G is at most $\prod_{i=0}^{j-1} ((d-i)!)^{n/(2(d-i))}$ and at least $\prod_{i=0}^{j-1} (2(d-i)/n)^{n/2}(n/2)!$. In particular, the number of Hamiltonian circuits of G is at most $(d!)^{n/(2d)}((d-1)!)^{n/(2(d-1))}/2$.*

Proof. The result follows on the bound on the permanent of Brègman [1] and Clark, George, and Potter [3] mentioned in the introduction, $(*)$ and $(**)$, for G and the successive sugraphs obtained from G by removing perfect matchings $M_1, \ldots, M_k$ one at a time, thus successively reducing the degree by one. For Hamiltonian circuits, the bound follows by choosing M_1 and M_2 forming the circuit in the two possible orders and dividing by two. $\square$

We may ask whether there exists in general an isomorphism of G sending M to M' for a Hamiltonian circuit C. We answer this in the case of the hypercube.

Theorem 4 *An isomorphism sending M to M' for a Hamiltonian circuit C in the d-dimensional hypercube exists only if $d = 2$.*

Proof. It is clear that such an isomorphism of the 2-dimensional hypercube mapping one perfect matching to the other exists. Suppose $d \geq 3$, and choose the $r = 2^{d-1} = 2k$ labels $1, 2, \ldots, r$ by labeling two vertices that differ only in the first dimension the same. Of p and p', one is odd and the other one even. If the isomorphism exists, we may write $p' = qpq'$, where q and q' define the isomorphism. The isomorphism given by q and q' is a composition of two types of isomorphisms, either flipping bit i or exchanging bits i and j, as every permutation of dimensions is a product of transpositions (see e.g. [8]). The case $d = 3$ can be verified directly as the Hamiltonian circuit is in that case essentially unique. If $d \geq 4$, then each subcube determined by dimensions $1, i$ in the case of a flip, or by dimensions $1, i, j$ in the case of an exchange, involves some number $r \leq 3$ of dimensions and t transpositions, for a total of $2^{d-r}t$ transpositions, which is even. For instance, if we flip dimension $i > 1$, we incur 2^{d-1} transpositions for both q and q'. Thus q and q' are both even, so p and $p' = qpq'$ are either both even or both odd, a contradiction to the fact that p and p' have different parity. $\square$

4 Number of Hamiltonian Circuits in Products by a Square

Theorem 5 *Let $G = Q_2 \Box G'$ be the product of a graph G' that has an even number of vertices and a Hamiltonian path, and a square Q_2 (the 2-dimensional hypercube). Then any perfect matching M of G that does not use the edges of Q_2 can be extended to a Hamiltonian circuit of G. Thus if $M_{G'}$ is the number of perfect matchings of G' and H is the number of Hamiltonian circuits of G, then $H \geq (M_{G'})^4$.*

Proof. We proceed in four steps (1),(2),(3), and (4). Let $G'_{00}, G'_{01}, G'_{10}, G'_{11}$ be the four copies of G' in G, and let $M_{00}, M_{01}, M_{10}, M_{11}$ be the corresponding perfect matchings forming M. Let M_0 be the union of M_{00} and M_{01} in G', and let M_1 be the union of M_{10} and M_{11} in G'. Each component of M_i can be viewed as an alternating cycle combining alternating edges of M_{i0} and M_{i1}. Thus (1) if we include all the edges that traverse dimension 2 in G in addition to the four perfect matchings M_{ij}, we have decomposed the vertices of G into parts that form induced cycles that take dimension 2 alternatingly.

It remains to combine these cycles into a single cycle. Let M' be the union of M_0 and M_1 in G', so that each vertex of M' corresponds to four vertices of the four M_{ij}. (2) Each component of M consists in G of cycles corresponding to components in M_0 and in M_1. These cycles can be pairwise combined by replacing at a shared vertex of two cycles corresponding to M_0 and to M_1 the two edges across dimension 2 by two edges across dimension 1 of Q_2. Thus if a vertex v of G' corresponding to vertices $v_{00}, v_{01}, v_{10}, v_{11}$ has edges $v_{00}v_{01}$ in a cycle from M_0 and $v_{10}v_{11}$ in a cycle from M_1 across dimension 2, we may remove these two edges and add edges $v_{00}v_{10}$ and $v_{01}v_{11}$ to combine the two cycles into a single cycle. We repeat this combining of cycles until each component of M' gives a single cycle in G.

Thus at this point each connected component of M' gives a cycle in G through the four vertices in G for each vertex in M'. These four vertices are joined either by the two edges across dimension 2 or by the two edges across dimension 1. (3) Notice that given a case of a single vertex v in a component of M' where the four vertices in G are joined across dimenension 2, we could start by replacing the two edges across dimension 2 with edges across dimension 1, or replace somewhere else, as each cycle in M_0 and in M_1 has an even number of vertices, so the final combination across dimension 1 can be done in an even number of vertices v of M', and thus in at least two such v.

(4) It remains to combine the resulting cycles corresponding to components of M' into a single cycle. Join the components of M' with a minimal number of edges from the Hamiltonian path of G' to form a single component in G'. This gives a tree-like structure to the components of M if we view these components as single vertices. Any component C of M has at some vertex u of C at most two child components C_1 and C_2 at vertices v and w respectively. If at u of M' we are using dimension 1 (resp. 2) of Q_2 for C, that is, the edges $u_{00}u_{10}$ and $u_{01}u_{11}$ were chosen for the four corresponding vertices u_{ij} of G, we may use dimension 1 (resp. 2) at C_1 and C_2 as well. Removing edges $u_{00}u_{10}$, $u_{01}u_{11}$, $v_{00}v_{10}$, $w_{01}w_{11}$, breaks C in two places and breaks each of C_1, C_2 in one place. Adding edges $u_{00}v_{00}$, $u_{10}v_{10}$, $u_{01}w_{01}$, $u_{11}w_{11}$, combines the cycles for the children C_1, C_2 connected at v, w respectively to the cycle for the parent C connected at u by the Hamiltonian path of G', forming the Hamiltonian circuit of G. Clearly we have $M_{G'}^4$ choices of possible M_{ij}, which proves $H \geq M_{G'}^4$. □

Corollary 2 *Let $G' = (U, V, E)$ be a regular bipartite graph, and let $G = Q_2 \Box G'$ be an n-vertex, d-regular graph that is the product of G' by a 2-dimensional cube Q_2. Suppose that G' has a Hamiltonian path. The number of Hamiltonian circuits of G is at least $((d/e)(1-o(1)))^{n/2}$ as d and n tend to infinity.*

Proof. Follows from Theorem 5 the remarks in the Introduction, formula $(**)$. We replace $1+o(1)$ by $1-o(1)$ because of the loss of degree 2 from d to $d-2$ since the square Q_2 is not used by the matching. □

The upper and lower bounds for Hamiltonian circuits in the hypercube differ essentially by a square (a factor of two in the exponent). We can reduce this gap by considering decompositions into cycles instead of Hamiltonian circuits, where the cycles are required to be of length a multiple of 2^k for some k.

Theorem 6 *The number of decompositions of the d-dimensional hypercube into cycles of length a multiple of 2^k is at least $((d/(ek))(1-o(1)))^n$.*

Proof. Partition the d dimensions into k groups R_i of $\lfloor d/k \rfloor$ or $\rceil d/k \lceil$ dimensions. If we combine together the dimensions in each group R_i and replace it by the parity of the bits in the group

R_i, we obtain a k-dimensional hypercube that has a Hamiltonian cycle C of length 2^k. Back in the original hypercube, each edge in the reduced cycle C corresponds to choosing matchings in smaller cubes corresponding to the $\lfloor d/k \rfloor$ or $\rceil d/k \rceil$ dimensions of each cube in one group R_i of dimensions, where by the remarks in the Introduction, formula (**), the number of choices per vertex is $(d/(ek))(1 - o(1))$. Combining these choices of matchings of subcubes over all n vertices gives the stated bound. $\square$

We now combine the approach of Theorems 5 and 6 to infer the following.

Theorem 7 *Let $G = Q_2 \Box G_1 \Box \cdots \Box G_k$ be an n-vertex graph that is the Cartesian product of a square Q_2 and k regular bipartite graphs G_i of degrees $d_i \geq f$ that have a Hamiltonian circuit, for k even. Then G has at least $((f/e)(1 + o(1)))^{n(1-k/2^k)}$ Hamiltonian circuits as f and k tend to infinity.*

Proof. If we do not take into account Q_2 and replace each bipartite graph G_i by two adjacent vertices v_0^i and v_1^i representing both sides of the bipartition, we obtain a k-dimensional hypercube that has a Hamiltonian cycle K that takes edges corresponding to the first dimension $v_0^1 v_1^1$ in alternation (this is also called a Gray code sequence). We shall consider dimension 2 of Q_2 in combination with the first graph G_1 as a single factor G_i. For each occurrence of an edge $v_0^i v_1^i$ in K we may take a perfect matching in G_i, so by the remarks in the Introduction, formulas (*) and (**), we have $((f/e)(1 + o(1)))^n$ possible choices of matchings that give a decomposition of G into cycles, as n vertices have each f/e choices to be matched in the appropriate G_i.

The proof is in four steps (1), (2), (3), and (4) as in the proof of Theorem 5. (1) Now for each choice of v_j^i, $i \geq 2$, that chooses v_0^i for all but at most one of the $i \geq 2$, replace the edge corresponding to (v_0^1, v_1^1) by the edge on dimension 1 of Q_2. This corresponds to k choices of the 2^k edges of K, namely all the $k - 1$ choices to choose a single v_1^i for $2 \leq i \leq k$ and the choice that chooses all v_0^i. Therefore the bound on the number of choices is reduced to $((f/e)(1 + o(1)))^{n(1-k/2^k)}$. Consider a choice M of such edges, which is decomposed into M_0 and M_1 as dimension 1 of Q_2 is not used so far. Thus as we traverse the cycle K we obtain paths in G that combine into cycles, and the vertices of G are decomposed into parts that form cycles of M, and thus unions of cycles M_0 and M_1. Now all cycles in such a choice go through dimension 2 of Q_2 with the choice v_0^i for $i \geq 2$.

We thus have a union of cycles corresponding to M_0 and to M_1 in both sides of dimension 1, and all such cycles go through vertices that choose v_0^i for all the $i \geq 2$. (2) We may combine such cycles by alternating dimensions 1 and 2 of Q_2 at the vertices that choose v_0^i for all but at most one of the $i \geq 2$ correspond to four vertices $v_{00}, v_{01}, v_{10}, v_{11}$, by removing edges $v_{00}v_{01}, v_{10}v_{11}$ and adding edges $v_{00}v_{10}, v_{01}v_{11}$.

(3) We may choose to exchange dimension 2 for dimension 1 at a given $v_{00}, v_{01}, v_{10}, v_{11}$, to join cycles or not, for each set of cycles that get combined in step (2), because k is even, and so choosing at most one v_1^i happens in an even number of choices of such four vertices both for M_0 and for M_1, so the number of choices of where to combine to complete the joining of cycles is thus even and therefore at least two different such choices of which four vertices where to combine exist.

(4) We shall show that a Cartesian product of cycles is Hamiltonian. Select the Hamiltonian cycle C_1 for G_1, and for each Hamiltonian cycle C_i for $i \geq 2$ consider the cycle C_i' going through the vertices of the form v_0^i by replacing paths of length two in C_i by a single edge in C_i'. The product $C_1 \Box C_2' \Box \cdots \Box C_k'$ has a Hamiltonian cycle C going through all vertices that choose v_0^i for each $i \geq 2$. We may replace each edge of C in dimension $i \geq 2$ by a path of length two by going back from C_i' to C_i, thus visiting some vertices with at most one v_1^i chosen for a cycle L. Select a

minimal number of edges from L to combine the cycles in step (3) into a tree-like structure, where the vertices of the tree correspond to cycles from step (3). We finally combine all the cycles from step (3) by using the chosen edges from C as follows. If vertex u in a cycle C has chosen children v and w by chosen edges of L in the tree-like structure, where these children form cycles C_1 and C_2 for v and w respectively, ignoring the square Q_2, then we proceed as follows. First we remove edges $u_{00}u_{10}$, $u_{01}u_{11}$, $v_{00}v_{10}$, $w_{01}w_{11}$, to break C in two places and break each of C_1 and C_2 in one place, across dimension 1 (similarly for dimension 2, with a matching choice of dimension 1 or 2 for C_1 and C_2 with regards to C by step (3)). Then we add edges $u_{00}v_{00}$, $u_{10}v_{10}$, $u_{01}w_{01}$, $u_{11}w_{11}$, thus combining the cycles for the children C_1, C_2 connected at v, w respectively to the cycle for the parent C connected at u.

This completes the construction of the Hamiltonian cycle for G, except for the technical lemma that claimes that the product of cycles $C_1 \Box C_2' \Box \cdots \Box C_k'$ has a Hamiltonian cycle C'. It suffices to iteratively replace a product of two cycles by a single cycle, i.e., to show that a product $C_1 \Box C_2$ of two cycles has a Hamiltonian cycle. If C_0 is $a_1 \cdots a_r$ and C_1 is $b_1 \cdots b_s$, we may take the cycle that starts $a_1b_1, a_1b_2, \ldots, a_1b_s$, then goes in alternating directions $a_2b_s, a_3b_s, \ldots, a_rb_s$, then $a_rb_{s-1}, a_{r-1}b_{s-1}, \ldots, a_2b_{s-1}$, then $a_2b_{s-2}, a_3b_{s-2}, \ldots, a_rb_{s-2}$, and so on, until it finishes with $a_rb_1, a_{r-1}b_1, \ldots, a_2b_1$ or $a_2b_1, a_3b_1, \ldots, a_rb_1$ back to a_1b_1. □

Corollary 3 *Let G be the n-vertex d-dimensional hypercube, with $n = 2^d$, and H_d be the number of Hamiltonian cycles of G. Then $((d \log 2/(e \log\log d))(1-o(1)))^n \le H_d \le (d!)^{n/(2d)}((d-1)!)^{n/(2(d-1))}/2$. So $H_d = M_d^{2-o(1)}$, that is, the number of Hamiltonian circuits in the hypercube is nearly quadratic in the number of perfect matchings.*

Proof. The upper bound is from Theorem 3. For the lower bound, we apply Theorem 7 with $f = \lfloor (d-2)/k \rfloor$ and choose k such that $2^k/k^2 = \log d$.

The bound with $n/2$ in the exponent from $(*)$ and $(**)$ for matchings is thus replace by a bound that is assymptotically n in the exponent for Hamiltonian circuits, thus showing that going from matchings to Hamiltonian circuits essentially squares M_d to H_d. □

5 Matchings and Hamiltonian Circuits in Grids

Theorem 8 *Let G be an n-vertex d-dimensional grid, which is the Cartesian product of paths $P_1, P_2, \ldots, P_d$, where P_i has $r_i \ge 2$ vertices, with r_1 even. When d tends to infinity, the graph G has at least $((d/(2e))(1-o(1)))^{n/2}$ perfect matchings and at most $((2d)!)^{n/(4d)}$ perfect matchings.*

Proof. The upper bound follows from the remarks in the Introduction, formulas $(*)$ and $(**)$, and the bound $2d$ on the degree.

For the lower bound, divide each path P_i of length r_i into $r_i' = \lfloor r_i/2 \rfloor$ matched pairs and at most one single additional vertex. This divides the grid into hypercubes of various dimensions. Suppose the r_i for $1 \le i \le k$ are even and the r_i for $k+1 \le i \le d$ are odd. Suppose we divide the $d-k$ odd dimensions into t dimensions for which we choose the r_i' matched edges and $d-k-t$ dimensions for which we choose the additional vertex. This gives cubes of $k+t$ dimensions with $(((k+t)/e)(1-o(1)))^{2^{k+t}/2}$ perfect matchings, and the number of such cubes is the product of $k+t$ factors r_i'. When we multiply these terms, the exponents add up to terms in the product of terms $2r_i'$ for $i \le k$ and terms $2r_i'+1$ for $i \ge k+1$, divided by 2, and this product is the product of the r_i divided by 2, which is $n/2$. This expression is significantly dominated by the terms with $k+t \geq d(1-o(1))/2$, giving the expression $(d(1-o(1))/(2e)^{n/2}$. □

Theorem 9 *Let G be the d-dimensional grid, the Cartesian product of paths $P_1, P_2, \ldots, P_d$, where P_i has $r_i \geq 2$ vertices. If $d=1$, or all r_i are odd, then G does not have a Hamiltonian circuit. Otherwise ($d \geq 2$ and some r_i is even) G has a Hamiltonian circuit.*

Proof. If $d=1$ then G is a path P_1 and does not have a Hamiltonian circuit. If all P_i have an odd number of vertices r_i, then G has an odd number of vertices $r = r_1 r_2 \cdots r_d$. A bipartite graph with an odd number of vertices cannot have a Hamiltonian circuit.

Suppose instead $d \geq 2$ and some P_i has r_i even, say P_1 has r_1 even. If $d=2$, then a Hamiltonian circuit is obtained by going down P_1 at the left end of P_2, then going in the direction of backwards P_1 one vertex at a time, each time traversing P_2 back and forth while avoinding the left vertex of P_2 that was already visited. Since r_1 is even, the last time P_2 will be traversed backwards to its leftmost vertex where the circuit was started. If $d \geq 3$, assume inductively the result without P_d for the product G' of $P_1, P_2, \ldots, P_{d-1}$, giving a Hamiltonian circuit C'. Place the even edges of C' at one end of P_d and the odd edges of C' at the other end of P_d in G, and add all copies of the path P_d to obtain the Hamiltonian circuit. □

Theorem 10 *Let G be a d-dimensional grid that has a Hamiltonian circuit as in Theorem 9. The number of Hamiltonian circuits of G is at least $((d \log 2/(2e \log\log d))(1-o(1)))^n$ and at most $((2d)!)^{n/(4d)}((2d-1)!)^{n/(4d-2)}/2$. when d tends to infinity.*

Proof. The upper bound follows by the remarks in the Introduction, formulas (∗) and (∗∗), by choosing a matching in a graph of degree at most $2d$, removing it, and choosing a matching in a graph of degree at most $2d-1$.

For the lower bound, decompose the grid into cubes as in Theorem 8, find Hamiltonian circuits in each subcube, with the asymptotics larger than for matchings by Corollary 3. To interconnect these subcubes, we contract the subcubes, replacing r_i by $\lceil r_i/2 \rceil$, while keeping r_1 even as r_i, and use on this smaller grid the cycle from Theorem 9. This requires entering and exiting each subcube at adjacent vertices x and y that have the edge (x, y) in the Hamiltonian cycle for the subcube. We choose x and y with all coordinates even except for at most two coordinates, and say x with an even number of odd coordinates. If the subcube must be exited in two dimensions in the direction that has an odd value in the dimension, then this determines the two odd value dimensions for x and the one odd value dimension for y. After exiting at such an x, there is one odd value dimension carried over from x into the next subcube, until we exit through an even value dimension, in which case the corresponding x will have all even values. This completes combining the cycles of the various subcubes, with the bounds following from Corollary 3 as in Theorem 8. □

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