Hamiltonian Cycles for Fixed Distance Graphs of the Hypercube

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1 Introduction and Results

We consider the n-dimensional hypercube H_n and for $0 \leq d \leq n$ the graph H_n^d whose vertices are the vertices of H_n and whose edges join vertices at distance d in H_n. Clearly H_n^0 is a graph with 2^n loops, and H_n^n is a graph matching antipodals. We thus restrict our attention to the more complex cases for $1 \leq d \leq n-1$. If d is odd, we show that H_n^d has a Hamiltonian cycle, and if d is even, we show that H_n^d has two connected components, one for odd and one for even vertices, and both components have a Hamiltonian cycle. Finally, we consider H_n' whose vertices are the vertices of H_n and whose edges join vertices at any distance $2 \leq d \leq n$, namely the complement of H_n, and show that this graph has maximum clique size 2^{n-1} and the only cliques of this size are the even and the odd vertices.

Lemma 1 *If n is odd, then H_n has a Hamiltonian path whose endpoints are antipodal vertices (at distance n from each other). If n is even, then H_n has a Hamiltonian path whose endpoints are almost antipodal vertices (at distance $n-1$ from each other), and H_n has a Hamiltonian cycle with antipodal vertices opposite to each other in the cycle.*

Proof. If $n = 1$, the single edge joins antipodal vertices. Suppose inductively the result holds for n.

If n is odd, then we consider the path given for n from v to the antipodal w, then switch dimension $n+1$ to get a vertex x at distance $n+1$ from v, and finally a Grey code gives a Hamiltonian cycle and thus a Hamiltonian path with switched dimension $n+1$ from x to y adjacent to x and thus at distance $n = (n+1)-1$ from v, thus v, y are almost antipodal.

If n is even, then we consider the path given for n from v to the almost antipodal w at distance $n-1$ from v, then switch dimension $n+1$ to get vertex x at distance n from v, and finally a Grey code gives a Hamiltonian cycle and thus a Hamiltonian path with switched dimension $n+1$ from x to y adjacent to x at distance $n+1$ from v, thus v, y are antipodal.

We can then construct a Hamiltonian path in H_{n-1} joining antipodal vertices v, w, then switch dimension n to get to a neighbor x of w such that x is antipodal to v, then traverse with switched dimension n the path antipodal to the path from x to v to return to x. This gives a Hamiltonian cycle in H_n such that opposite vertices in the cycle are antipodal to each other, and at even distance from each other in the cycle. □

Lemma 2 *If n is odd, then H_{n+1}^n has a Hamiltonian cycle.*

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Proof. From the previous lemma, replacing every vertex at even distance from x in the cycle with its antipodal flips half of the vertices in the Hamiltonian cycle of H_{n+1} with their antipodals and gives a Hamiltonaian cycle in H_{n+1}^n. We notice as well that the previous lemma is not actually needed, as flipping antipodals would happen in any Hamiltonian cycle, since antipodals are at even distance in the cycle because the dimension $n+1$ is even. □

Theorem 1 *Let $1 \leq d \leq n-1$. If d is odd, then H_n^d has a Hamiltonian cycle, and if d is even, then the even and odd vertices of H_n^d each give a connected component with a Hamiltonian cycle.*

Proof. If $n = d+1$ then the result is the preceeding Lemma. We suppose inductively that the result holds for d, n, and show the result for $d, n+1$. Let uv be an edge in a Hamiltonian cycle in H_n, flip d positions including position $n+1$ to get from u, v to x, y at distance d from u, v, so that uv, xy, ux, vy are all edges of H_{n+1}^d, then consider a Hamiltonian cycle in H_n with position $n+1$ flipped and going through xy, which exists by symmetry of the edges of H_n^d. Removing uv, xy from both cycles and adding ux, vy gives the Hamiltonian cycle in H_{n+1}^d. This completes the proof for d odd.

If d is even, do the construction for $d-1, n-1$ above with $d-1$ odd, then add the position n for each vertex of the cycle as the parity of the first $n-1$ positions, which will give a Hamiltonian cycle of the even vertices of H_n^d. For the odd vertices, just flip position n. □

Theorem 2 *The maximum clique of H_n' is of size 2^{n-1} and must consist of either the even or the odd vertices of H_n'.*

Proof. Equivalently, consider maximum independent sets in H_n. A perfect matching of H_n has size 2^{n-1}, and only one vertex per edge of the matching can be chosen, giving the upper bound 2^{n-1}. This can only be reached if each edge of the matching is covered by the independent set as well, and since each edge of H_n belongs to a perfect matching, then independent set must be a vertex cover as well.

Then if x is in the independent set I which is also a vertex cover, then the neighbors of x cannot be in I, so the vertices at distance 2 from x must be in I, the vertices at distance 3 from x must not be in I, the vertices at distance 4 from x must be in I, and so on. Thus I is precisely the vertices at even distance from x, thus either the even or the odd vertices. □

Theorem 3 *Let G be a Cartesian product of two cliques K^1 and K^2 at least one of each is of size at least 3. Let G' be the complement of G, namely all edges in G' join vertices that differ in both K^1 and K^2. Then G' has a Hamiltonian cycle.*

Proof. If say K^1 has size two, then G' is a complete bipartite graph with with $n \geq 3$ vertices in each side minus a perfect matching, and this graph is clearly Hamiltonian by advancing by two from one side to the other and retreating by one on the way back.

If instead $|K^i| = ab^i \geq 3$, where $a = \gcd(|K^1|, |K^2|)$, then denote the vertices of K^i as $0, 1, 2, \ldots, ab^i - 1$ modulo ab^i, and consider two systems of cycles S^1 advancing by $(1,1)$ of length ab^1b^2 for a typical cycle C^1, and S^2 advancing by (b^1, b^2) of length a for a typical cycle C^2. If $a = 1$, then C_1 is a Hamiltonian cycle, while if $b_1 = b_2 = 1$, then instead we let S^2 advance by $(1,2)$.

The union of the two S^i is the Cartesian product of the two C^i, which is Hamiltonian. Indeed, if say C^1 is of even length, then advance the whole length minus one of C^2, then one edge of C^1, then return the whole length minus one of C^2, then one edge of C^1 in the same direction as before, then repeat the process until the Hamiltonian cycle is closed. If both C^1 and C^2 are of odd length at least 3 then retreat the whole length minus one of C^1 just once, then do the previous process but with the whole length minus two of C^2, and close the cycle finally with advancing whole length minus one of C^2, giving a Hamiltonian cycle. $\square$

We generalize partially to more factors.

Theorem 4 *Let G be a Cartesian product of m cliques K^i which are either (1) all of the same size at least 3, or (2) of pairwise relatively prime sizes, or (3) of sizes divisible by p^2 for some prime p. Let G' join vertices that differ in all K^i. Then G' has a Hamiltonian cycle.*

Proof. If the sizes of K^i are pairwise relatively prime, then advancing as before by $(1, 1, \ldots, 1)$ gives us a Hamiltonian cycle like C_1.

If the size of K^i are all the same, then we consider as before m systems of cycles generated by $(1, 1, 1, \ldots, 1, 1)$,$(1, 2, 1, \ldots, 1, 1)$, $(1, 1, 2, \ldots, 1, 1)$,...,$(1, 1, 1, \ldots, 2, 1)$,$(1, 1, 1, \ldots, 1, 2)$. forming a Cartesian product of m cycles which can be combined two at a time to obtain a Hamiltonian cycle as before.

If all sizes of K^i are divisible by p^2, let $a_i = |K^i|/p$, and consider for clique K_i the cycle of length $|K^i|$ generated by $(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_m)$. Multiplying this by p gives $(0, \ldots, 0, p, 0, \ldots, 0)$, so all multiples of p in all p positions can be obtained in combination. Proceeding modulo p, we then obtain $(0, \ldots, 0, 1, 0, \ldots, 0)$, since p divides a_j. Thus these m cycles give a Cartesian product of m cycles which can be combined two at a time to obtain a Hamiltonian cycle as before. $\square$

We generalize partially to fewer flips.

Theorem 5 *Let G be a Cartesian product of m cliques K^i which are either (1) all of the same size at least 3, or (2) of pairwise relatively prime sizes, or (3) of sizes divisible by p^2 for some prime p. Let G' join vertices that differ in r of the m cliques K^i, and suppose r divides m only in case (2). Then G' has a Hamiltonian cycle.*

Proof. If the sizes of K^i are pairwise relatively prime, and r divides m, then divide the m factors into m/r groups of r, and for each choice of such a group, consider advancing by flipping the r positions in the group. This gives a Cartesian product of m/r cycles that has a Hamiltonian cycle as before.

If the size of K^i are all the same, then we consider as before m systems of cycles generated, say in the case $r = 4$ and $m = 7$, by $(1, 1, 1, 1, 0, 0, 0)$,$(1, 2, 1, 1, 0, 0, 0)$, $(1, 1, 2, 1, 0, 0, 0)$,$(1, 1, 1, 2, 0, 0, 0)$, $(1, 1, 1, 0, 1, 0, 0)$,$(1, 1, 1, 0, 0, 1, 0)$,$(1, 1, 1, 0, 0, 0, 1)$, and in general first an initial block of r 1s, then change successively all but the first 1 into 2, then shift the last 1 of the group of 1s to the end. This forms a Cartesian product of m cycles which can be combined two at a time to obtain a Hamiltonian cycle as before.

If all sizes of K^i are divisible by p^2, then as before let $a_i = |K^i|/p$, and consider for clique K_i the cycle of length $|K^i|$ generated, instead of by $(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_m)$, by replacing $m - r$ of the a_j with 0s so that exactly r nonzero entries are botained. Thus these m cycles give a Cartesian product of m cycles which can be combined two at a time to obtain a Hamiltonian cycle as before. $\square$