List Homomorphisms of Graphs with Bounded Degrees

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August 8, 2003

Abstract

In a series of papers we have classified the complexity of list homomorphism problems. Here we investigate the effect of restricting the degrees of the input graphs. It turns out that the complexity does not change (except when the degree bound is two). We obtain similar results on restricting the size of the lists.

We contrast these results with facts about some variants of the list homomorphism problem, where restricting the degrees can have an important effect on the complexity.

1 Introduction

We consider undirected graphs without multiple edges, but with loops allowed. A graph without loops is called *irreflexive*, and a graph in which each vertex has a loop is called *reflexive*. Note that a bipartite graph is, by definition, irreflexive.

A *homomorphism* $f : G \to H$ is a mapping $f : V(G) \to V(H)$ such that $f(g)f(g')$ is an edge of H for each edge gg' of G. Every graph H gives rise to a decision problem HOM_H in which one is to decide whether or not a given input graph G admits a homomorphism to the fixed graph H. It is shown in [16] that each of these problems HOM_H is polynomial-time solvable (if H is bipartite or contains a loop), or NP-complete (if H is irreflexive and nonbipartite). A number of generalizations of this basic family of problems has been considered, and in [17] the authors have set up a framework that contains all these variants.

In particular, when the input graphs G are restricted to have degrees bounded by Δ, we obtain the family of problems HOM_H^{Δ}. We shall assume throughout the rest of the paper that $\Delta \geq 3$, since graphs with degrees bounded by 2 are unions of paths and cycles, and all the problems are polynomial-time solvable by easy or standard techniques, cf. [6, 20]. Even when $\Delta \geq 3$ restricting the degrees can have a significant impact on the complexity of the problem. For instance, it is well known [15] that the problem HOM_H with $H = K_3$ (the problem of 3-colourability) is NP-complete, while the problem HOM_H^{Δ} is polynomial-time solvable when $\Delta = 3$, since, by the theorem of Brooks [5], a connected graph with maximum degree three is either 3-colourable or isomorphic to K_4. On the other hand [14], when $\Delta \geq 4$, the problem HOM_H^{Δ} is once again NP-complete. In [14], there are more complex examples of this phenomenon (where hard problems

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become easy when a low enough degree bound is imposed). It is also shown there that when H is an odd cycle of length at least five, the problem HOM_H^Δ remains NP-complete for all $\Delta \geq 3$.

For each fixed graph H, the *list homomorphism problem* LHOM_H asks whether or not a graph G together with lists, $L(g) \subseteq V(H), g \in V(G)$, admits a homomorphism f of G to H with $f(g) \in L(g)$ for all $g \in V(G)$. (Such a homomorphism will be called a *list homomorphism* with respect to the lists L.) In a series of papers [9, 10, 11], we have obtained the following classification of the complexity of the problems LHOM_H:

Theorem 1 *[11] The problem $LHOM_H$ is polynomial-time solvable when H is a bi-arc graph, and is NP-complete otherwise.*

Bi-arc graphs are defined as follows: Let C be a fixed circle, with two chosen points n and s. A *bi-arc* is an ordered pair of arcs (N, S) on C such that N contains n but not s, and S contains s but not n. A graph H is a *bi-arc graph* if there exists a family of bi-arcs $(N_h, S_h), h \in V(H)$, such that for any $h, h' \in V(H)$ one of the following two alternatives must happen: Either h and h' are not adjacent in H, N_h intersects $S_{h'}$, and $N_{h'}$ intersects S_h; or h and h' are adjacent in H, N_h does not intersect $S_{h'}$, and $N_{h'}$ does not intersect S_h. (In both cases $h = h'$ is possible.)

It turns out, cf. [11], that a reflexive graph is a bi-arc graph if and only if it is an *interval graph*, and an irreflexive graph is a bi-arc graph if and only if it is bipartite and its complement is a *circular arc graph*. In this sense, bi-arc graphs provide a common generalization of these well studied graph classes.

In fact, we have a transformation which changes an arbitrary graph H into a bipartite graph H^* in such a way that H is a bi-arc graph if and only if the complement of H^* is a circular arc graph, [11]. The graph H^* is called the *associated bipartite graph* of the graph H, and is defined to have the vertex set $\{n_h, s_h : h \in V(H)\}$ and the edge set $\{n_h s_{h'}, s_h n_{h'} : hh' \in E(H)\}$. It is immediate from the definitions that a bi-arc representation of H is a circular arc representation of the complement of H^*. It is shown in [11] that if LHOM_{H^*} is NP-complete then LHOM_H is also NP-complete.

In [17], the authors introduced the following family of problems. Given a graph H and an integer Δ, the problem LHOM_H^Δ is the restriction of the problem LHOM_H to inputs G with all degrees less than or equal to Δ. In the first version of [17], the authors observed that, according to the above theorem, the problem LHOM_H^Δ is polynomial time solvable when H is a bi-arc graph, and posed as an open problem the question of classifying the complexity of LHOM_H^Δ for other graphs. We solve this problem by showing that all other problems LHOM_H^Δ are NP-complete, i.e., that the complexity of list homomorphisms does not change when degree constraints are imposed. (The final version of [17] cites this result.) Thus our main result in this paper is the following classification theorem:

Theorem 2 *Let $\Delta \geq 3$ be fixed. The problem $LHOM_H^\Delta$ is polynomial-time solvable when H is a bi-arc graph, and is NP-complete otherwise.*

We also investigate the effect of restricting the size of the lists of the input graphs G. When all lists have size at most two, any list H-colouring problem can be solved using 2-satisfiability. For any $k \geq 3$, we shall show that the list H-colouring problem restricted to graphs with all lists of size at most k is still NP-complete unless H is a bi-arc graph. In fact, we show that w can impose the degree constraint and the list size constraint at the same time, without changing the complexity classification.

Three restrictions of the list homomorphism problem LHOM_H are of interest:

In the *extension problem* EXT_H, the inputs are restricted to have lists which are either singletons $\{v\}$, or the entire set $V(H)$. (Thus the extension problem asks whether or not a given partial mapping - a *pre-colouring* - can be extended to a homomorphism. In the case of classical colourings, this 'pre-colouring extension problem' has been studied by many authors [1, 18, 19]. If there are no degree restrictions, the extension problem is equivalent to the so called 'retraction' problem, cf. [9].)

In the *connected list homomorphism problem* CLHOM_H, the inputs G are restricted to have lists $L(g) \subseteq V(H), g \in V(G)$, each of which induces a connected subgraph of H. (Note that EXT_H is a restriction of CLHOM_H, as long as H is a connected graph.)

In the *degree-list homomorphism problem* DLHOM_H, the inputs G are restricted to have lists L satisfying $|L(g)| \geq deg_G(g)$ for each vertex g of G.

The degree-restricted versions EXT_H^Δ and CLHOM_H^Δ of the first two problems are defined in the obvious way, restricting the inputs further to have maximum degrees at most Δ.

It is clear that if H is connected, then EXT_H^Δ is a restriction of the problem CLHOM_H^Δ, and that each CLHOM_H^Δ a restriction of the problem LHOM_H^Δ. When $\Delta \leq |V(H)|$, there is a simple polynomial-time reduction of EXT_H^Δ to the problem DLHOM_H - each pre-coloured vertex g of the input G is replaced by $deg_G(g)$ vertices, all pre-coloured by the same colour. Finally, it is also clear that DLHOM_H is a restriction of LHOM_H.

In [12], we investigate the complexity of the problems CLHOM_H^Δ, EXT_H^Δ, and DLHOM_H, when H is a reflexive or irreflexive cycle. Interestingly, in these situations restricting the degree can have an important effect on the complexity of the problem. We obtain several new polynomial algorithms, with corresponding Brooks-type characterizations. As a byproduct we obtain general results classifying the complexity of connected list homomorphism problems CLHOM_H^Δ.

The following two tables summarize some of results of [12], showing the graduation of complexity of these homomorphism problems (to irreflexive and reflexive cycles H) in terms of the degree bounds:

Table I: Irreflexive cycles

MAXIMUM DEGREE Δ	LENGTH k	HOM	EXT	CLHOM	LHOM
$\Delta = 3$	$k = 3$	P	P	NP-c	NP-c
	$k = 4$	P	P	P	P
	$k \geq 5$ odd	NP-c	NP-c	NP-c	NP-c
	$k = 6$	P	P	NP-c	NP-c
	$k \geq 8$ even	P	NP-c	NP-c	NP-c
$\Delta \geq 4$	$k = 3$	NP-c	NP-c	NP-c	NP-c
	$k = 4$	P	P	P	P
	$k \geq 5$ odd	NP-c	NP-c	NP-c	NP-c
	$k = 6$	P	P	NP-c	NP-c
	$k \geq 8$ even	P	NP-c	NP-c	NP-c

Table II: Reflexive cycles

MAXIMUM DEGREES Δ	LENGTHS k	HOM	EXT	CLHOM	LHOM
$\Delta = 3$	$k = 3$	P	P	P	P
	$k = 4, 5$	P	P	NP-c	NP-c
	$k \geq 6$	P	NP-c	NP-c	NP-c
$\Delta = 4$	$k = 3$	P	P	P	P
	$k = 4$	P	P	NP-c	NP-c
	$k \geq 5$	P	NP-c	NP-c	NP-c
$\Delta \geq 5$	$k = 3$	P	P	P	P
	$k \geq 4$	P	NP-c	NP-c	NP-c

The problems DLHOM$_H$ are particularly interesting for graphs H in which every vertex v is adjacent to all but at most one vertex (possibly itself). For instance, when H is the irreflexive complete graph K_n, the problem DLHOM$_H$ has a polynomial time algorithm via the characterization theorem due to Erdös, Rubin, and Taylor [8]. In the companion paper we give a new proof of this result, one that we found easier to adapt to other graphs H with the above property, also discussed there.

In [9], the first two authors proved that, for a reflexive graph H, the problem CLHOM$_H$ is polynomial-time solvable when H is a chordal graph (i.e., contains no induced cycle of length greater than three), and is NP-complete otherwise. Using the above results for reflexive cycles we see that the complexity of these problems does not change when degree constraints are imposed:

Theorem 3 *[12] Let $\Delta \geq 3$ and a reflexive graph H be fixed. The problem CLHOM$_H^\Delta$ is polynomial-time solvable when H is a chordal graph, and is NP-complete otherwise.*

In [17], the authors consider other variants of these problems - in particular variants in which not only the existence of homomorphisms is to be decided, but also their numbers counted. (This is a problem of interest in statistical physics, cf. [3, 7, 17].) Since we have, in this paper, fewer variants of the problems, we use a somewhat simpler notation than that of [17].

2 Proof of Theorem 2

When H is a bi-arc graph, the polynomial-time solvability of LHOM$_H^\Delta$ follows from Theorem 1 and the fact that LHOM$_H^\Delta$ is a restriction of LHOM$_H$. It follows from the proof of Theorem 1 (cf. [11]) that if LHOM$_{H^*}^\Delta$ is NP-complete then so is LHOM$_H^\Delta$, for any Δ. Thus we only need to show that that LHOM$_H^\Delta$ is NP-complete if H is a bipartite graph whose complement $\overline{H}$ is not a circular arc graph.

Let H be a bipartite graph. Recall that the definition of a bipartite graph implies that the graph is irreflexive. Since a graph which has a loop does not admit a homomorphism to an irreflexive graph, we shall assume that all graphs in this section are irreflexive. For a similar reason we may restrict the input graphs G also to be bipartite. In fact, we may assume that the vertices of G and H are coloured black and white, and that black vertices of G have lists consisting of black vertices of H, and white vertices of G have lists consisting of white vertices of H (cf. [10]).

One structure which prevents a bipartite graph H from being the complement of a circular arc graph is the present of an edge-asteroid [10]:

An *edge-asteroid* A *of order* $2k+1$ ($k \geq 1$) in a bipartite graph H (with white and black vertices) is

- a set of $2k+1$ edges $e_0 = 00', e_1 = 11', \dots, e_{2k} = (2k)(2k)'$ of H where each i is black and each i' white, together with
- a set of $2k+1$ paths, $P_0, P_1, \dots, P_{2k}$ in H, such that, for each i, the path P_i joins i to $i+1$, and $P_i \cup \{i', (i+1)'\}$ does not contain any neighbours of $i+k+1$ or $(i+k+1)'$. (Subscripts are modulo $2k+1$.)

An edge-asteroid in which none of the sets $P_1 \cup \{1', 2'\}, P_2 \cup \{2', 3'\}, \dots, P_{2k-1} \cup \{(2k-1)', (2k)'\}$ contain any neighbours of 0 or $0'$ is called *special*.

Proposition 1 *If $\Delta \geq 3$ and H contains a special edge-asteroid, then the problem LHOM$^{\Delta}_{H}$ is NP-complete.*

Proof. Assume that H has a special edge-asteroid A with edges $00', 11', \dots, (2k)(2k)'$, and paths $P_0, P_1, \dots, P_{2k}$, as defined above. In [10] we have shown how to reduce the problem of 3-colourability to LHOM$_H$ using gadgets called *choosers*. We take this opportunity to simplify and streamline the proof given in [10] (as well as to correct some details); then we modify the construction to apply to LHOM$^{\Delta}_{H}$, for any $\Delta \geq 3$.

Let P be a graph with a specified *starting vertex* s and a specified *ending vertex* t, in which each vertex v has a list $L(v) \subseteq V(H)$. Suppose furthermore, that for each $\ell \in L(s)$ we are given a set $S(\ell) \subseteq L(t)$. Members of the set $S(\ell)$ are called the *outputs associated with the input* ℓ. Then P is a *chooser* with the *input set* $L(s)$ and the *output sets* $S(\ell), \ell \in L(s)$, provided

- each list homomorphism $f : P \to H$ with respect to L has $f(t) \in S(f(s))$, and
- for any input $\ell \in L(s)$ and any of its associated outputs $r \in S(\ell)$, there exists a list homomorphism $f : P \to H$ with respect to L in which $f(s) = \ell, f(t) = r$.

We shall first construct the following basic choosers.

Lemma 1 *For each $i = 0, 1, \dots, 2k$, there is*

- *a chooser B_i, with input set $\{i, i+k\}$ and output sets $S(i) = \{i\}$, $S(i+k) = \{i+k+1\}$,*
- *a chooser B'_i, with input set $\{i, i+k+1\}$ and output sets $S(i) = \{i\}$, $S(i+k+1) = \{i+k\}$,*
- *a chooser B^{+}_i, with input set $\{i, i+k\}$ and output sets $S(i) = \{i\}$, $S(i+k) = \{i+k, i+k+1\}$,*
- *a chooser B'^{+}_i, with input set $\{i, i+k+1\}$ and output sets $S(i) = \{i\}$, $S(i+k+1) = \{i+k, i+k+1\}$,*
- *a chooser B^{++}_i, with input set $\{i, i+k, i+k+1\}$ and output sets $S(i) = \{i\}$, $S(i+k) = S(i+k+1) = \{i+k+1\}$, and*

- *a chooser $B_i'^{++}$, with input set $\{i, i+k, i+k+1\}$ and output sets $S(i) = \{i\}$, $S(i+k) = S(i+k+1) = \{i+k\}$,*

Proof. In fact, we can take each chooser to be a path, with the starting and ending vertices being the two endpoints of the path.

Consider the path $P_{i+k}: i+k = x_1, x_2, \ldots, x_q = i+k+1$ from $i+k$ to $i+k+1$ in A. Note that q must be odd, since both $i+k$ and $i+k+1$ are black. We take B_i to be a path $y_1, y_2, \ldots, y_q$ of the same odd length q, with $s = y_1, t = y_q$, and lists $L(y_r) = \{i, x_r\}$ when r is odd, and $L(y_r) = \{i', x_r\}$ when r is even. The inputs of this B_i are the elements of $L(y_1)$, i.e., i and $i+k$. Any list homomorphism of B_i which maps y_1 to i must map the consecutive vertices $y_2, y_3, y_4, \ldots, y_q$ to $i', i, i', \ldots, i$ respectively, since neither i nor i' has neighbours in P_{i+k}. Similarly, any list homomorphism of B_i which maps y_1 to $i+k$ must map the consecutive vertices $y_2, y_3, \ldots, y_q$ to $x_2, x_3, \ldots, x_q = i+k+1$. Moreover, the mappings so described are list homomorphisms. Thus B_i is a chooser which associates the output i with the input i and the output $i+k+1$ with the input $i+k$, as claimed.

Only small modifications are needed for the other five choosers B_i', B_i^+, $B_i'^+$, B_i^{++}, $B_i'^{++}$, and the arguments are very similar. In B_i, if we exchange s and t, i.e., we let $s = y_q$ and $t = y_1$, to obtain a chooser B_i'. A chooser B_i^+ is obtained from B_i by changing the lists to $L(y_r) = \{i, x_r, i+k\}$ when r is odd, and to $L(y_r) = \{i', x_r, (i+k)'\}$ when r is even. Note that this does not change the set of inputs $L(s) = L(y_1) = \{i, i+k\}$ since $x_1 = i+k$. It is easy to verify, using the properties of the edge-asteroid, that this chooser B_i^+ associates output i with input i, and outputs $i+k, i+k+1$ with the input $i+k$. Similarly, we obtain a chooser $B_i'^+$ from B_i' by changing the lists to $L(y_r) = \{i, x_r, i+k+1\}$ when r is odd, and $L(y_r) = \{i', x_r, (i+k+1)'\}$ when r is even. A chooser B_i^{++} is obtained from B_i by changing the lists to $L(y_r) = \{i, x_r, i+k+1\}$ when r is odd, and $L(y_r) = \{i', x_r, (i+k+1)'\}$ when r is even. Finally, we obtain a chooser $B_i'^{++}$ from B_i' by changing the lists to $L(y_r) = \{i, x_r, i+k\}$ when r is odd, and $L(y_r) = \{i', x_r, (i+k)'\}$ when r is even. $\square$

We also need the following special choosers, which take advantage of the special position of the edge $00'$ in A.

Lemma 2

- *For $i \neq k, k+1$, there is a chooser D_i with input set $\{0, i, i+k\}$ and output sets $S(0) = \{0\}$, $S(i) = \{i\}$ and $S(i+k) = \{i+k+1\}$.*
- *For $i \neq k, k+1$, there is a chooser D_i' with input set $\{0, i, i+k+1\}$ and output sets $S(0) = \{0\}$, $S(i) = \{i\}$ and $S(i+k+1) = \{i+k\}$.*
- *For each $i, j \neq 0$, there is a chooser $D_{i,j}$ with input set $\{0, i\}$ and output sets $S(0) = \{0\}$, $S(i) = \{j\}$.*
- *For each $i, j \neq 0$, there is a chooser $D_{i,j}^+$ with input set $\{0, i, j\}$ and output sets $S(0) = \{0\}$, $S(i) = S(j) = \{j\}$.*
- *For each $i, j \neq 0$, there is a chooser $D_{i,j}^{++}$ with input set $\{0, i\}$ and output sets $S(0) = \{0\}$, $S(i) = \{i, j\}$.*

Proof. We shall again construct choosers that are paths. In fact, all of these choosers can be obtained by modifying the choosers from Lemma 1. A chooser D_i from B_i by changing the lists to $L(y_r) = \{0, i, x_r\}$ when r is odd, and $L(y_r) = \{0', i', x_r\}$ when r is even. Since $i \neq k, k+1$, neither 0 nor $0'$ has neighbours in P_{i+k}. Hence the only output of 0 is 0, and the outputs for i and $i+k$ remain the same as in B_i. Similarly, we obtain a chooser D'_i from B'_i by changing the lists to $L(y_r) = \{0, i, x_r\}$ when r is odd, and $L(y_r) = \{0', i', x_r\}$ when r is even.

We next show how to construct a chooser $D_{i,j}$. Let $P_{i,j} : \ i = x_1, x_2, \ldots, x_q = j$ be the concatenation of the paths $P_i, P_{i+1}, \ldots, P_{j-1}$ if $i < j$, or the concatenation of the reversed paths $P_{i-1}^{-1}, P_{i-2}^{-1}, \ldots, P_j^{-1}$ if $i > j$. Notice that in either case q is again odd, and the concatenated paths contain only vertices not adjacent to either 0 or $0'$, since our edge-asteroid is special. The chooser $D_{i,j}$ will be a path of the same (odd) length q, $s = y_1, y_2, \ldots, y_q = t$, and lists $L(y_r) = \{0, x_r\}$ when r is odd, and $L(y_r) = \{0', x_r\}$ when r is even. Since the above concatenated paths contain no neighbours of either 0 nor $0'$, $D_{i,j}$ is a chooser that associates output 0 with input 0 and output j with input i.

Finally, both choosers $D^+_{i,j}$ and $D^{++}_{i,j}$ can be obtained from $D_{i,j}$ by modifying the lists: To obtain $D^+ i, j$, we let $L(y_r) = \{0, x_r, j\}$ when r is odd, and $L(y_r) = \{0', x_r, j'\}$ when r is even and, to obtain $D^{++}_{i,j}$, we let $L(y_r) = \{0, x_r, i\}$ when r is odd, and $L(y_r) = \{0', x_r, i'\}$ when r is even. $\square$

Lemma 3 *Denote $\ell = k+1$. There exists a chooser*

- *C_1 with input set $\{0, 1, \ell\}$ and output sets $S(0) = \{1\}$ and $S(1) = S(\ell) = \{1, \ell\}$,*
- *C_2 with input set $\{0, 1, \ell\}$ and output sets $S(0) = \{\ell\}$ and $S(1) = S(\ell) = \{1, \ell\}$,*
- *C_3 with input set $\{0, 1, \ell\}$ and output sets $S(1) = \{\ell\}$ and $S(0) = S(\ell) = \{0, \ell\}$,*
- *C_4 with input set $\{0, 1, \ell\}$ and output sets $S(1) = \{0\}$ and $S(0) = S(\ell) = \{0, \ell\}$,*
- *C_5 with input set $\{0, 1, \ell\}$ and output sets $S(\ell) = \{0\}$ and $S(0) = S(1) = \{0, 1\}$, and*
- *C_6 with input set $\{0, 1, \ell\}$ and output sets $S(\ell) = \{1\}$ and $S(0) = S(1) = \{0, 1\}$.*

Proof. All these choosers can be found by concatenating the basic choosers from Lemmas 1 and 2:

$$\begin{aligned}
C_1 &= D^+_{1,k+1} D^{++}_{k+1,1} B^{++}_{k+1}. \\
C_2 &= D^+_{1,k+1} D_{k+1,k} D^{++}_{k,2k} B'^{++}_k B'_{2k} B'_{k-1} B'_{2k-1} \ldots B'_{k+2} B'_1. \\
C_3 &= D_1 D_{k+2} D_2 D_{k+3} \ldots D_{k-1} D_{2k} B^{++}_k B'^{+}_k D^+_{2k,k} B_0. \\
C_4 &= D_1 D_{k+2} D_2 D_{k+3} \ldots D_{k-1} D_{2k} B^{++}_k B'^{+}_k D'_{2k} D'_{k-1} D'_{2k-1} \ldots D'_1 B'^{++}_{k+1}. \\
C_5 &= B^{++}_{k+1} B'^{+}_{k+1} D_1 D_{k+2} D_2 D_{k+3} \ldots D_{2k} B^{++}_k D_{k,1}. \\
C_6 &= B^{++}_{k+1} B'^{+}_{k+1} D^+_{k+1,1}.
\end{aligned}$$

We leave the straightforward verifications to the reader. $\square$

Given a graph G, we have shown in [10] how to use the six choosers in Lemma 3 to construct, in polynomial time, a graph G' with lists $L(v), v \in V(G')$, such that G is 3-colourable if and only if G' admits a list homomorphism to H, with respect to the lists L. Moreover,

- $V(G) \subseteq V(G')$,

- each vertex of $V(G') - V(G)$ has degree at most two, and
- each vertex of $V(G)$ has the list $\{0, 1, \ell\}$.

The degrees in G' of the vertices in $V(G)$ may be large. (In fact, each $v \in V(G)$ has $deg_{G'}(v) = 3deg_G(v)$.) We replace each vertex v of G of degree d in G' by a cycle C_v of length $2d$, with consecutive vertices having lists $\{0, 1, \ell\}, \{0', 1', \ell'\}, \{0, 1, \ell\}, \ldots \{0', 1', \ell'\}$. We now connect the d distinct edges incident with v to the d distinct vertices of C_v with lists $\{0, 1, \ell\}$, as in the proof of Proposition 2. The resulting graph is called G''. Since the edges $00', 11', \ell\ell'$ have no other adjacencies amongst their endpoints (according to the definition of a special edge-asteroid), it follows that a list homomorphism of G'' to H (with respect to the lists L) maps each connecting vertex of C_v the same element of $\{0, 1, \ell\}$. Thus G' has a list homomorphism to H if and only if G'' has a list homomorphism to H (both with respect to the lists L). Since G'' has all degrees at most three, we have completed the proof of Proposition 1. □

According to [10], a bipartite graph is the complement of a circular arc graph if and only if it contains no induced (even) cycle of length at least six and no special edge-asteroids. It is easily seen that a cycle of length at least ten contains a special edge-asteroid of order three. Thus to complete the proof of Theorem 2 it will suffice to show that LHOM$_H^\Delta$ is NP-complete if H contains an induced cycle of length six or eight.

Proposition 2 *If $\Delta \geq 3$ and H contains an induced cycle of length six or eight, then the problem LHOM$_H^\Delta$ is NP-complete.*

Proof. We begin with the six-cycle. Let H contain the six-cycle with consecutive vertices $1, 2', 3, 1', 2, 3'$. We reduce the problem of two-colourability of three-uniform hypergraphs (also known as the the NOT-ALL-EQUAL-3-SATISFIABILITY problem without negated variables [15]). Given a three-uniform hypergraph F, we replace each vertex v of F by a six-cycle C_v, consisting of consecutive vertices $v(1, 2), v'(2', 3'), v(3, 1), v'(1', 2'), v(2, 3), v'(3', 1')$. Each vertex $v(i, j)$, has the list consisting of i and j; these lists are considered ordered, with i being the *first* element and j being the *second* element of the list. Similarly, each vertex $v'(i', j')$ has the list with the first element i' and the second element j'. Let $V(i, j)$ denote the set of all vertices $v(i, j)$ over all v in F. Finally, we replace each hyperedge uvw of F by a separate vertex $e(uvw)$ adjacent precisely to $u(1, 2), v(2, 3), w(3, 1)$, and having the list $\{1', 2', 3'\}$.

Calling the resulting graph G' we claim that F is two-colourable if and only if G' has a list homomorphism to H. Indeed, if the hypergraph F is colourable with colours $1, 2$, then we can map each vertex of the cycle C_v to the i-th element of its list, where i is the colour of the vertex v in F. Since no hyperedge is monochromatic, each vertex $e(uvw)$ can also be coloured. Conversely, suppose G' has a list homomorphism to H. It is easy to check that if all vertices of C_v must map to the first element of their list, or all to the second element of their list, for each v. If the vertex $e(uvw)$ of G' is mapped to i', then none of the vertices $u(1, 2), v(2, 3), w(3, 1)$ can be mapped to i; clearly, this means that not all can be mapped to the first element of their lists, and not all can be mapped to the second element of their lists.

It remains to make sure that all degrees are at most three. In the graph G' there may be vertices of higher degrees, but only in the sets $V(i, j)$. We replace each such vertex x of degree $d > 3$, with list $\{i, j\}$, by a $2d$-cycle C_x, with consecutive vertices having the lists $\{i, j\}, \{i', j'\}, \{i, j\}, \ldots, \{i', j'\}$, connecting the d distinct vertices with lists $\{i, j\}$ to the d distinct edges incident with x. The resulting graph will be called G''. It is clear that either all vertices with the list $\{i, j\}$ will be mapped to i, or all will be mapped to j. Therefore G' admist a list homomorphism to H if and only if G'' admits such a list homomorphism. Moreover, G'' has all vertices of degree at most three.

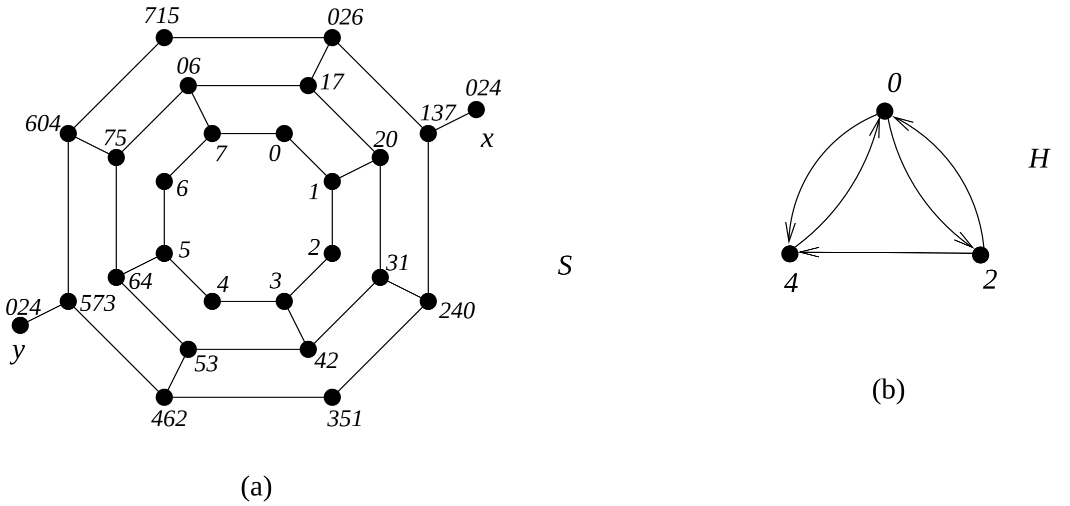

Figure 1: The gadget (a) used for reduction from H-colouring (b)

For the eight-cycle we reduce the problem from the homomorphism problem for the digraph H in Fig. 1 (b). It is known that the H-colouring problem is NP-complete [2]. Thus suppose G is a digraph, and let G' be the digraph obtained by replacing each arc xy of G with a separate copy of the gadget S, with lists as indicated, from Fig. 1 (a). It is not hard to check that the vertices x, y of S can take on precisely the pairs of values $0, 2$ or $0, 4$, or $2, 0$, or $2, 4$, or $4, 0$, respectively. (The innermost eight-cycle is mapped in a unique way due to its lists; each subsequent eight-cycle can then rotate either clockwise or counterclockwise. In particular, if a vertex on the outermost eight-cycle maps to the i-th member of its list, $i = 1, 2, 3$, then all vertices of the outermost eight-cycle map to the i-th member of their list.) Therefore G is homomorphic to H if and only if G' admits a list colouring by the eight-cycle. The graph G' has degrees at most three, except at the vertices of G, where the degrees may be higher, and where the lists are $0, 2, 4$. We now replace each such vertex v by the graph $Y(v)$ consisting of three concentric cycles of length $8d$, where d is the degree of v in G', with adjacencies and lists as indicated in Fig. 2. (The innermost cycle has lists $0, 1, 2, \ldots, 7, 0, 1, \ldots$, and in general, all the cycles have lists that repeat with period eight.) The outermost cycle has d vertices of degree two with the lists $0, 2, 4$, which are easily seen to have to take on the same value (0 or 2 or 4) for any list homomorphism to the eight-cycle. They can therefore be used to attach the d edges leading to v, without creating degrees greater than 3. In other words, if G'' is obtained from G' by replacing each vertex of degree greater than three with its own $Y(v)$, then G'' has degrees at most three and admits a list homomorphism to the eight-cycle if and only if G' has such a list homomorphism. □

We now observe that in all the proofs we have never used lists of size greater than three, and conclude:

Corollary 1 *The list H-colouring problem restricted to instances G with maximum degree three and all lists of size at most three is NP-complete, unless H is a bi-arc graph.*

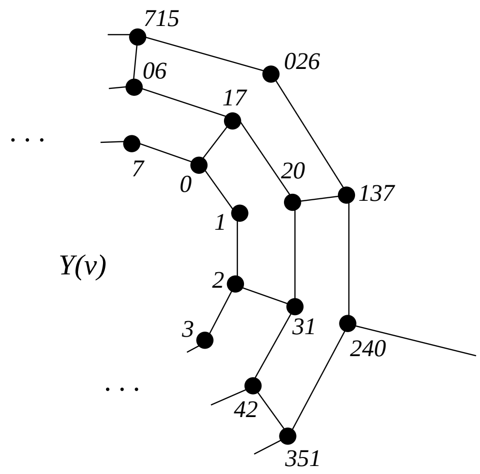

Figure 2: The replacement digraph for v

Based on the above arguments, we make the following meta-conjecture:

Conjecture 1 *Let H be fixed. Any variant of the H-colouring problem (including list homomorphisms, and homomorphisms of more general structures) which is NP-complete without degree constraints is also NP-complete with degree constraints, provided the degree bound is high enough.*

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