Channel Assignment in Wireless Networks and Classification of Minimum Graph Homomorphism

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Abstract

We study the problem of assigning different communication channels to access points in a wireless Local Area Network. Each access point will be assigned a specific radio frequency channel. Since channels with similar frequencies interfere, it is desirable to assign far-apart channels (frequencies) to nearby access points. Our goal is to assign the channels so as to minimize the overall interference experienced by all access points. The above problem can be formulated as an instance of the Minimum Graph Homomorphism problem. We give a complete description of all possible approximation classes for the general formulation of the problem.

1 Introduction

In this paper, we consider the problem of assigning communication channels to access points in a wireless Local Area Network (LAN). Each communication channel uses a distinct radio frequency. However, frequencies within the same region of frequency spectrum or band will interfere with one another. Because of this reason, it is desirable to assign far-apart channels (frequencies) to nearby access points. Motivated by this, we formalize the MIN INTERFERENCE problem as follows: Number the k available channels in increasing order of radio frequency: $1, \ldots, k$. Given the location of all n access points, we'll build a proximity graph as follows: each vertex represent an access point, and two potentially interfering access points are connected by an edge. We use a simple yet practical interference cost model as follows: Each channel i incurs an interference cost of 1 with itself, and an interference cost of 0.5 with adjacent channels $i-1$ or $i+1$ (should they exist), and 0 with others channels. Thus, if both end-points of an edge are assigned the same channel, the cost is 1; if they are assigned adjacent channels, the cost is 0.5; otherwise, the cost is 0. The goal is to assign the k channels to access points so as to minimize the overall interference cost.

The formulation of the above MIN INTERFERENCE problem is closely related to other well-studied NP-complete problems. For instance, it is closely related to graph coloring. Specifically, We can simplify the interference cost model as follows and obtain the MINIMUM EDGE DELETION k-PARTITION problem: Each channel i incurs an interference cost of 1 with itself, and 0 with the rest of the channels. The MINIMUM EDGE DELETION k-PARTITION problem is well studied. In the case of $k = 2$, the problem can be approximated within a factor of $O(\log n)$ in polynomial time as shown by Garg, Vazirani, and Yannakakis [4]. Recently, the approximation factor has been improved to $O(\sqrt{\log n})$ by Charikar et al. [3]. For the lower bound, the problem is NP-hard to approximate within a factor of $1 + \epsilon$ for some $\epsilon > 0$, even for instances that have an optimal

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solution of cost at least $\delta|E(G)|$, for some $\delta > 0$, as shown by Håstad [5]. In the case of $k \geq 3$, this problem is NP-hard to approximate within an additive $\epsilon|E(G)|$ for some $\epsilon > 0$, even for instances that have an optimal solution of zero cost, as shown by Petrank [7].

On the other hand, we may further generalize the MIN INTERFERENCE problem as follows. Let J be a fixed complete reflexive graph with weighted loops and edges. Let $0 \leq a_0 < a_1 < \cdots < a_k$ denote the set of weights, which are all constants. Given an irreflexive graph G, the goal of the the MIN GRAPH HOMOMORPHISM problem is to provide a mapping f from $V(G)$ to $V(J)$ with the minimum cost. The cost of a mapping is the sum of the weights of the edges $(f(u), f(v))$ in J over all edges (u, v) in G. The MIN INTERFERENCE problem is the special case where G is the proximity graph on access points and the graph J is a path on k vertices with self-loops of weight 1 on each vertex, while the weight of the path edges is 0.5. Alon et al. show that the MIN GRAPH HOMOMORPHISM problem can be solved within an additive $\epsilon \cdot m$ in time $2^{\tilde{O}(n^2/(\epsilon^2 m))}O(n^2)$ for $m = |E(G)|$ [1].

The main results of the paper are approximation and hardness results for the MIN GRAPH HOMOMORPHISM problem. Depending on the parameters $a_0, a_1, \ldots, a_k$, we show that the problem varies from trivial to additively $\Omega(E|G|)$ hard. We cover all possible cases of $a_0, a_1, \ldots, a_k$. The rest of the paper is organized as follows: Section 2 provides the overview of the classification, ranging from easy to hard. Section 3 provides detailed proofs to two key theorems.

2 Classification of MIN GRAPH HOMOMORPHISM

We begin the classification with the easy cases. If there exists a self-loop on node u with weight a_0 in J, then the problem is trivial. We map all vertices in G to u, guaranteeing the minimum cost. Otherwise, there is no self-loop with the least weight.

If $a_0 > 0$, then the following algorithm gives a $((a_0 + a_k)/2a_0)$-approximation. In J, identify the edge (u, v) such that the weight of (u, v) is a_0. For each vertex in G, the algorithm randomly maps it to either u or v with equal probability. The algorithm can be derandomized easily as well. Thus we obtain a constant approximation. Note that this case is MAX SNP-hard, due to the hardness result of MINIMUM EDGE DELETION 2-PARTITION.

We are thus left with the case $a_0 = 0$, which we analyze depending on whether the edges of weight zero form a bipartite graph or not. We introduce the following two theorems, for which we give the detailed proofs in the next section.

Theorem 1 *If the edges of weight zero form a non-empty bipartite graph, then the* MIN GRAPH HOMOMORPHISM *problem is within a constant factor of the* MINIMUM EDGE DELETION 2-PARTITION *problem.*

Theorem 2 *If the edges of weight zero form a nonbipartite graph, then the* MIN GRAPH HOMOMORPHISM *problem is NP-hard to approximate within an additive factor $\epsilon \cdot m$ for some $\epsilon > 0$ even when the optimum solution has cost 0, and $|E(G)| = m \geq n^{2-\delta}$ for any $\delta > 0$. If $NP \not\subseteq O(2^{n^\mu})$ for some $\mu > 0$, then the problem cannot be approximated within $\epsilon \cdot m$ in time $O(2^{(n^2/m)^\delta})$ for some $\epsilon, \delta > 0$.*

Theorems 1 and 2 both give the same additive hardness of approximation, which applies to the case $a_0 > 0$ as well by subtracting a_0 from all weights. Note that we can always get an additive approximation factor of $a_k|E(G)|$, regardless of the values of $a_0, a_1, \ldots, a_k$, which will be constant factor away from the hardness result in the theorems above. Combining the results, we obtain the following classification.

Theorem 3 *The* MIN GRAPH HOMOMORPHISM *problem falls in four approximation classes:*

1. *Trivial: If there is a self-loop of least weight;*
2. *Constant factor easy, and constant factor hard (even when optimum is proportional to the number of edges): If there is no zero weight edge and no loop of least weight;*
3. *$O(\sqrt{\log n})$-easy, and constant factor hard (even when optimum is proportional to the number of edges): If the zero weight edges form a bipartite graph with at least one edge;*
4. *additively $O(|E(G)|)$-easy for dense graphs, and additively $\Omega(|E(G)|)$-hard (even when the optimum is zero for graphs that are not dense): If the zero weight edges form a nonbipartite graph without loops.*

3 Detailed Proofs for Theorem 1 and 2

Proof for Theorem 1: Let H be the graph with $V(H) = V(J)$ and $E(H)$ consisting of the edges of H of weight 0. The graph H is bipartite. Since there is a homomorphism from H to a graph consisting of two adjacent vertices, the problem of minimizing the number of edges of G that map to non-edges of H is the minimum edge deletion 2-partition problem, which can be approximated within $O(\log n)$ [4], or within $O(\sqrt{\log n})$ [3]. Thus our problem can be approximated within $O((a_k/a_1)\log n)$, or within $O((a_k/a_1)\sqrt{\log n})$.

We prove hardness of approximation by induction on the size of J. Let J' be obtained by removing from J all vertices not incident to any edge of weight 0. If J' has fewer vertices than J, then apply the inductive hypothesis to J' for instances G'. Such an instance G' can be transformed into an equivalent instance G for J, by attaching to each vertex v in G', of degree $d(v)$, a total of $n(v) = \lceil d(v)a_k/a_1 \rceil$ edges joining v to $n(v)$ new vertices. An optimal solution for G can be assumed to assign v to a vertex in J', otherwise the $n(v)$ added edges would incur cost at least $d(v)a_k$ instead of 0, and the $d(v)$ neighbors of v in G' incur cost at most $d(v)a_k$.

We may thus assume that every vertex in J is incident to some edge of weight 0. Let r be the number of vertices in J, and suppose J has vertex u incident to two edges uv and uw of weight 0. Let J^p be the graph obtained from J by assigning to each edge xy in J^p the minimum total weight of a path of length $2r+1$ from x to y in J, where the path may traverse loops and visit the same vertex multiple times. A path of odd length greater than $2r+1$ in J can be made of length $2r+1$ by removing an even cycle (since some vertex appears three times on the original path) and then traversing 0 length edges in both directions. In J^p, for the two vertices v, w, the weight of an edge vx is the same as the weight of the corresponding edge wx. Thus the problem for J^p is equivalent to the problem for $J'' - w$. We apply the inductive hypothesis to $J'' - w$ on instances G', and to obtain an equivalent instance G for J, we replace each edge of G' with a path of length $2r+1$.

We may thus assume that every vertex in J is incident to exactly one edge of weight 0. Let the *mate* of a vertex x in J be the vertex y such that xy has weight 0. The graph J^p in the preceding construction has the property that if x, y are mates, and x', y' are mates, then the edges xx' and yy' have the same weight. We show hardness of approximation for J^p, which implies hardness of approximation for J as before. We may consider instances G that have multiple parallel edges and multiple loops, by reducing such instances to equivalent instances G' without parallel edges or loops, as follows. If a vertex v in G has $p(v)$ parallel edges and $\ell(v)$ loops, replace v by vertices $v_i, 1 \le i \le p(v) + 2\ell(v)$, one for each edge incident to v, and join each v_i, $i > 1$ to v_1 by $\lceil a_k/a_1 \rceil$ paths of length 2. These paths will incur weight 0 if and only if v_i maps to the same vertex in J^p, otherwise the path incurs weight at least a_k. We may thus change a solution by assigning v_i to the same vertex as for v_1, since this saves at least a_k on the paths of length 2 and incurs cost at most a_k on the single additional edge incident to v_i.

We may assume that there exists an instance G_0 for which every optimal solution uses every vertex in J^p, otherwise the problem is reduced to a proper subgraph of J^p. Let s_0 be the optimal cost for G_0, and let $t_0 > s_0$ be the minimum cost for G_0 over all suboptimal solutions. If J^p has mates x, y with loops of weight a_1, then every optimal solution uses only x, y, since all other mates x', y' may be assigned to x, y. Otherwise J^p has four distinct vertices x, y, x', y' such that x, y are mates, x', y' are mates, and the edges xx' and yy' have weight a_1. Let a_i be the weight of edges xy' and yx'. Two cases arise: (1) If the loops at x, y, x', y' have weight $a_1 + a_i$, let $\hat{J}^p$ be the subgraph of J^p induced by x and y'; the problem on $\hat{J}^p$ is hard to approximate within an additive $\delta|E(G)|$ for some $\delta > 0$ by hardness of approximation of minimum edge deletion 2-partition [5]. In this case let $s_1 = a_1, t_1 = 2a_1, x_0 = x', y_0 = y$. (2) Otherwise, say the loops at x and y have weight $b < a_1 + a_i$, and let $\hat{J}^p$ be the subgraph of J^p induced by the vertices z that give the minimum sum s_1 of the weights of xz and yz, where $\hat{J}^p$ consists of a collection of pairs of mates not including x', y'; the problem on $\hat{J}^p$ is hard to approximate within an additive $\delta|E(G)|$ for some $\delta > 0$ by inductive hypothesis. Let $t_1 > s_1$ be the next smallest sum of weights of xz and yz over vertices z in J^p, after s_1, and let $x_0 = x, y_0 = y$ in this case.

Given an instance G' for $\hat{J}^p$, we construct an instance G for J. Suppose G_0 has q vertices. Then for each pair of vertices p_i in G_0, we include in G a copy G'_i of G', and we also include in G a copy of G_0. If a vertex v in G'_i has degree $d(v)$, then we include in G additional $\lceil d(v)a_k/(t_1 - s_1)\rceil$ parallel edges joining v to both vertices of p_i. So far G has at most $q^2(2 + a_k/(t_1 - s_1))|E(G')|$ edges. We replace each edge in the copy of G_0 in G by $2q^2(2 + a_k/(t_1 - s_1))|E(G')|a_k/(t_0 - s_0)$ parallel edges, for a total of $c|E(G')|$ edges, where the constant c depends on $q, a_k, s_0, t_0, s_1, t_1$. The number of parallel edges in the copy of G_0 guarantees that if G_0 is assigned with cost t_0 or more, instead of s_0, then an additive cost of at least $\delta'|E(G)|$ will be incurred with respect to the optimal solution, for some constant $\delta' > 0$. We may thus assume that the cost for the copy of G_0 is s_0 per parallel edge in the solution found, and thus G_0 uses every vertex of J^p. In particular some pair p_i in the copy of G_0 maps to (x_0, y_0), and the parallel edges joining G_i to G_0 guarantee that the vertices in G_i will be assigned to vertices in $\hat{J}^p$. The solution found for G_i incurs an additional $\delta|E(G')| = \delta''|E(G)|$ cost by inductive hypothesis, for some constant $\delta'' > 0$. □

Proof for Theorem 2: Let H be the graph with $V(H) = V(J)$ and $E(H)$ consisting of the edges of H of weight 0. The *H-coloring problem* asks whether there is a homomorphism from G to H, and is NP-complete whenever H is nonbipartite, as shown by Hell and Nešetřil [6]. We follow the proof of this result given by Bulatov [2]. First assume that H is a *core*, that is, there is no homomorphism of H into a proper subgraph of H. If H is not a core, then H contains a subgraph H' that is a core and such that there is a homomorphism from H to H'. If H is a core, then we may use vertices of H in an instance G, by including a copy of H in G, since this copy of H must map to H by an isomorphism. It is shown in [2] that there exist two vertices $u_0 \neq u_1$ in H such that given any Boolean relation $R(x_1, \ldots, x_t)$, there exists a graph T containing H as a subgraph, and containing special distinct vertices $v_1, \ldots, v_t$, such that there are homomorphisms f from T to H leaving the subgraph H of T fixed, with the property that (1) each such f satisfies $f(v_i) \in \{u_0, u_1\}$, and (2) the possible tuples $(f(v_1), \ldots, f(v_t))$ are the tuples $(u_{x_1}, \ldots, u_{x_t})$ such that the relation $R(x_1, \ldots, x_t)$ holds. The problem is thus NP-complete by the NP-completeness of Boolean 3-satisfiability.

To reduce an instance G' of the minimum edge deletion 3-partition problem to our problem, we represent the three parts for the 3-partition by three Boolean assignments 001, 010 and 100. Consider the triangle on these three parts, giving as its edge relation the Boolean relation $R(x_1, x_2,$

$x_3, y_1, y_2, y_3)$ with tuples $(001, 010), (001, 100), (010, 001), (010, 100), (100, 001), (100, 010)$. A copy of the corresponding graph T, containing a copy of H as described above, can then be used as a gadget for each edge of the instance G'. Thus, by the hardness of approximation of minimum edge deletion 3-partition [7], we infer that there exists a $\delta > 0$ such that it is NP-hard to distinguish graphs G that have a homomorphism to H from graphs G such that every mapping from $V(G)$ to $V(H)$ has at least $\epsilon|E(G)|$ edges of G that map to non-edges of H, thus showing that the problem is hard to approximate within an additive $\epsilon a_1|E(G)|$ even when the optimum has cost 0. We may increase $|E(G)|$ by making n/k copies of the k vertices for $k = n^\delta$ and $k = n^2/m$. □

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References

[1] N. Alon, W. Fernandez de la Vega, R. Kannan, and M. Karpinski, Random sampling and approximation of MAX-CSP problems, J. Comput. Syst. Sci. 67(2) (2003) 212–243.

[2] A.A. Bulatov, H-coloring dichotomy revisited, Theoretical Computer Science, to appear.

[3] M. Cherikar et al., personal communication.

[4] N. Garg, V.V. Vazirani, and M. Yannakakis, Approximate max-flow min-(multi)cut theorems and their applications, SIAM J. Comp. 25 (1996) 235–251.

[5] J. Håstad, Some optimal inapproximability results, Proc. 29th Annual ACM Symp. on Theory of Computing, ACM (1997) 1–10.

[6] P. Hell and J. Nešetřil, On the complexity of H-coloring, J. of Combinatorial Theory, Ser. B, 48 (1990) 92–110.

[7] E. Petrank, The hardness of approximation: gap location, Computational Complexity, 4 (1994) 133-157.