SOME NOTES ON NUF DIGRAPHS

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Abstract. For the moment, we outline how to characterise undirected, irreflexive graphs that admit a near-unanimity operation of arity at most k using the notion of dual of a tree; the same is done for reflexive graphs, and for directed bipartite graphs. We hope to complete this by showing that the generating sets for the varieties in question contain only chordal extensible graphs.

1. Disclaimer

These notes are intended for a mature audience (i.e. this is not a draft but a set of notes for collaborators !) The symbol ♠ indicates a claim or statement I don't want to forget to verify/clarify later.

2. Introduction

(terms in italics will be defined later in the text)

Our goal is to give as nice a description as possible of various types of graphs that admit near-unanimity operations. We shall consider undirected reflexive graphs and *bipartite digraphs*; undirected irreflexive graphs will also be dealt with as a byproduct of our analysis of the directed case.

In the directed case, we describe an explicit family of digraphs that generate the variety of all bipartite digraphs that admit a near-unanimity operation (henceforth nuf) of arity k. It is our hope to prove various results on these digraphs, for instance to give a simple explicit description for $k = 4$ (we have $k = 3$; this was also known before) and to prove that for every k the digraphs are *chordal extensible.*

Here's a brief outline of the succession of ideas, all proofs and definitions will follow.

- if $\overrightarrow{\mathbb{H}}$ is a bipartite digraph that admits an nuf of arity k, then so does the structure $\widehat{\mathbb{H}}$ obtained by adding all singleton unary relations to $\overrightarrow{\mathbb{H}}$; this structure is clearly a core.
- Because $\widehat{\mathbb{H}}$ admits an nuf it has finite duality;[1]
- let $\{T_1, \dots, T_s\}$ be a complete set of (tree) obstructions for $\widehat{\mathbb{H}}$. Then
  - the number of *coloured elements* of each T_i is at most $k-1$, and

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[1]**Warning.** This, of course, does not hold for all types of structures; it happens to work for reflexive, undirected graphs and also for directed bipartite graphs. More on this topic later hopefully. ♠

  - $\widehat{\mathbb{H}}$ is the core of the product of the *duals* $D(T_i)$, in particular, the digraph $\overrightarrow{\mathbb{H}}$ is a retract of the product of the (underlying digraphs of the) $D(T_i)$.
- Consequently, *the (underlying digraphs of the) duals of trees with at most $k-1$ coloured elements form a generating set for the variety of bipartite digraphs admitting an nuf of arity k.*

Some remarks on the above:

(1) At first glance there is a problem with the above because the underlying digraphs of duals are not necessarily bipartite; however, this can be fixed easily. Indeed, the directed edge always appears as a dual, and hence we may take the product of each dual with this edge to obtain a generating set consisting of bipartite digraphs only (see details below)
(2) another problem seems to be: how do we know these duals have an nuf of the correct arity ? This follows from the fact we control the number of "coloured elements" of the tree obstructions.
(3) The exact same arguments work with reflexive, undirected graphs.
(4) For the case of undirected, irreflexive digraphs, we obtain a similar result even though existence of an nuf does NOT imply finite duality. The trick is this:
    - if $\mathbb{H}$ is an irreflexive, undirected graph with a k-ary nuf, then it is bipartite (in fact, it dismantles to an edge);
    - if $\overrightarrow{\mathbb{H}}$ is a digraph obtained from $\mathbb{H}$ by orienting every edge from one colour class towards the other, then $\mathbb{H}$ admits a k-ary nuf if and only if $\overrightarrow{\mathbb{H}}$ does.
    - Hence we may apply the result on directed bipartite graphs: $\overrightarrow{\mathbb{H}}$ is a retract of a product of duals; it is not hard to see that then $\mathbb{H}$ is a retract of a product of the undirected duals. Hence, the undirected versions of the generating digraphs will generate the variety of graphs with a k-ary nuf.

    Some other remarks:
    - Duals of trees, in general, can be quite large, and this could be a problem. However, any structure homomorphically equivalent will do. We describe some nice tricks below to cut down on the size of the duals.
    - In the irreflexive case, it is fairly straightforward to show that the duals of tree obstructions with 2 colours are actually paths. The same very certainly ♠ works for the reflexive case. Hence, we obtain as a nice, simple, special case of our approach results of Bandelt et al. (irreflexive) and of Pouzet et al. (reflexive) and a new analogous result for the irreflexive, directed bipartite case for **majority** (di)graphs.

3. The approach

We'll use blackboard fonts such as $\mathbb{G}$, $\mathbb{H}$, etc. to indicate relational structures and as usual G, H, etc. will indicate their respective universes. Let $\mathbb{H}$ be an undirected, irreflexive graph; if $\mathbb{H}$ is bipartite, let $\overrightarrow{\mathbb{H}}$ denote the digraph obtained from $\mathbb{H}$ by orienting every edge from one colour class towards the other (there are of course two possibilities, doesn't matter for now). More precisely, if D and U denote the two colour classes of $\mathbb{H}$, then we let $x \to y$ in $\overrightarrow{\mathbb{H}}$ precisely when $x \in D$ and $y \in U$ and xy is an edge of $\mathbb{H}$. A digraph obtained this way we will call a *bipartite digraph.* Finally, let $\widehat{\mathbb{H}}$ denote the structure obtained from $\overrightarrow{\mathbb{H}}$ by adding all singleton, unary relations. More precisely, it is the structure of type $\sigma' = \{E, S_h(h \in H)\}$ where $E(\widehat{\mathbb{H}})$ is the directed edge structure of $\overrightarrow{\mathbb{H}}$ and $S_h = \{h\}$ for all $h \in H$.

One remark:

Lemma 3.1. *Let $\mathbb{H}$ be a (di)graph. Then $\mathbb{H}$ admits an nuf of arity k iff each of its connected components does.*

Proof. A simple exercise, but see Lemma 3.3 below for a similar idea. □

The above remark allows us to consider only connected digraphs in what follows.

3.1. **Undirected, irreflexive graphs.** We start with a few simple observations concerning undirected, irreflexive graphs and their connection to the directed, bipartite case. A graph *dismantles to an edge* if by successively removing dominated vertices we eventually obtain an edge.

Lemma 3.2 ([2])**.** *Let $\mathbb{H}$ be a connected, undirected, irreflexive graph. If $\mathbb{H}$ admits an nuf then it dismantles to an edge; in particular it is bipartite.*

Remark. The fact that these graphs are bipartite might appear somewhere else ? If you want the proof of the lemma I can send you the notes.

Lemma 3.3. *Let $\mathbb{H}$ be a connected, undirected irreflexive bipartite graph. Then the following are equivalent:*

(1) *$\mathbb{H}$ admits an nuf of arity k;*
(2) *$\overrightarrow{\mathbb{H}}$ admits an nuf of arity k;*
(3) *$\widehat{\mathbb{H}}$ admits an nuf of arity k;*

Proof. Let D (Down) and U (Up) denote the colour classes of $\mathbb{H}$ such that $x \to y$ in $\overrightarrow{\mathbb{H}}$ implies that $x \in D$ and $y \in U$.

(2) ⇔ (3): immediate since nuf's are idempotent operations.

(1) ⇒ (2): Suppose that f is an nuf of arity k that preserves the edge structure of $\mathbb{H}$. Let $d \in D$ and $u \in U$; since $\mathbb{H}$ is connected there exists a large enough (even) integer N such that there exists a path of length N from any $x \in D$ to d, and similarly for U. Since f is idempotent it follows that $f(D^k) \subseteq D$ and $f(U^k) \subseteq U$. It is easy to conclude that f preserves the edge structure of $\overrightarrow{\mathbb{H}}$ also.

(2) ⇒ (1): Let f be an nuf of arity k that preserves the edge structure of $\overrightarrow{\mathbb{H}}$. We define a map $F : H^k \to H$ as follows: let $x = (x_1, \ldots, x_k) \in H^k$. If x is in the connected component of $\mathbb{H}^k$ containing the diagonal, then let $F(x) = f(x)$. If x is in a component of H^k where D or U appears $k-1$ times, let $F(x) = x_i$ where i is the smallest index in the repeated block. [More precisely, if $Y \subseteq W \times \cdots \times W \times Z \times W \times \cdots \times W$ where $\{W, Z\} = \{D, U\}$ and Z appears in the j-th position, then i is the smallest index different from j.] If x is in any other component, let $F(x) = x_1$.

We show that F is a homomorphism (it clearly is an nuf). Let xy be an edge of the component containing the diagonal. Then without loss of generality each entry of x is in D and each entry of y is in U. Hence $x \to y$ so $f(x) \to f(y)$ and thus $F(x)F(y)$ is an edge of $\mathbb{H}$. Since F is a projection on all other components it trivially preserves the edge structure there also. □

3.2. **Arity of nuf's and colours of obstructions.**

Lemma 3.4 ([1]). *If* $\widehat{\mathbb{H}}$ *admits an nuf then it has finite duality.*

Ramblings. It is interesting to note that the proof uses results about *poset* obstructions; it is easy to see however that posets never have finite duality (they do not have finitely many critical obstructions, but posets with nufs are precisely those with finitely critical obstructions that are themselves posets. This says that transitivity is somehow the only reason why posets with an nuf do not have finite duality (once you take the transitive closure of all obstructions of a poset with an nuf you obtain finitely many poset obstructions.))

Recall (see [3]): a structure $\mathbb{T}$ of the same type as $\widehat{\mathbb{H}}$ is an *obstruction for* $\widehat{\mathbb{H}}$ if there is no homomorphism $\mathbb{T} \longrightarrow \widehat{\mathbb{H}}$. An obstruction $\mathbb{T}$ is *critical* if every proper substructure of it admits a homomorphism to $\widehat{\mathbb{H}}$. A structure has finite duality if and only if it has finitely many critical obstructions; it is known that if this holds these obstructions are *trees* (see [3] for the definition of a tree in the case of general relational structures, or see below for one that is sufficient in our setting.)

Definition. We say that a vertex $x \in T$ is *coloured* if it belongs to a unary relation $S_h(\mathbb{T})$.

The following is an adaptation of a result of Zádori [6].

Lemma 3.5. *The structure* $\widehat{\mathbb{H}}$ *admits an nuf of arity* k *if and only if every of its critical obstructions has at most* $k-1$ *coloured elements.*

Proof. Suppose that $\widehat{\mathbb{H}}$ has an nuf f of arity k; then it has nufs of every arity larger than k (just add fictitious variables). Hence it will suffice to prove that $\widehat{\mathbb{H}}$ has no critical obstruction with k coloured elements. If $\mathbb{T}$ were such an obstruction, with vertices $t_i \in S_{h_i}(\mathbb{T})$ for $1 \leq i \leq k$, let $\mathbb{T}_i$ be the substructure obtained from $\mathbb{T}$ by removing the vertex t_i from the relation $S_{h_i}(\mathbb{T})$. Since $\mathbb{T}$ is critical we have homomorphisms $f_i : \mathbb{T}_i \longrightarrow \widehat{\mathbb{H}}$ for all $1 \leq i \leq k$. Define a map $\phi : T \longrightarrow H$ by $\phi(x) = f(f_1(x), \ldots, f_k(x))$. Obviously f preserves all relations, including the S_h since if $x = t_i$, then $f_j(x) = h_i$ for all $j \neq i$ and since f is an nuf it follows that $\phi(t_i) = h_i$.

Conversely, suppose that every critical obstruction of $\widehat{\mathbb{H}}$ has at most $k-1$ coloured elements. Consider the structure $\mathbb{G}$ of the same type as $\widehat{\mathbb{H}}$ which consists of the underlying digraph of $\widehat{\mathbb{H}}^k$ together with, for each $h \in H$, the unary relation $S_h(\mathbb{G})$ consisting of all tuples $(x_1, \ldots, x_k)$ where at least $k-1$ of the entries are equal to h. Obviously there exists a homomorphism from $\mathbb{G}$ to $\widehat{\mathbb{H}}$ if and only if $\widehat{\mathbb{H}}$ admits a k-ary nuf; if it does not then $\mathbb{G}$ contains a critical obstruction $\mathbb{T}$, and by hypothesis it has only $k-1$ coloured vertices. But if $x^i \in S_{h_i}$ for $1 \leq i \leq k-1$ are any $k-1$ coloured vertices in $\mathbb{G}$ then there exists a coordinate j such that the j-th coordinate of x^i is equal to h_i; hence the j-th projection would be a homomorphism from $\mathbb{T}$ to $\widehat{\mathbb{H}}$, a contradiction. □

Remark. The above is actually a fairly general result: given a structure $\mathbb{H}$ of type σ, consider the structure $\widehat{\mathbb{H}}$ of type σ' which is type σ plus the unary relations S_h. Then of course $\mathbb{H}$ and $\widehat{\mathbb{H}}$ have exactly the same idempotent operations, in particular nufs. The result above then says that $\mathbb{H}$ admits a k-ary nuf if and only if every critical obstruction of $\widehat{\mathbb{H}}$ has at most $k-1$ coloured elements. (It happens that for certain structures (such as those we consider in this paper) having a bounded number of coloured elements in the critical obstructions is sufficient to prove that the obstructions have also a bounded number of hyperedges, and hence finite duality.)

3.3. **Trees and duals.** What follows can be done in any language, but we'll assume that we are working, as above, in the language $\sigma' = \{E, S_h(h \in H)\}$ of $\widehat{\mathbb{H}}$ where $\mathbb{H}$ is some fixed graph with an nuf.

Our idea is to list all critical obstructions of $\widehat{\mathbb{H}}$; by Lemma 3.4 there are finitely many of these, say $\mathbb{T}_1, \dots, \mathbb{T}_s$ are the critical obstructions of $\widehat{\mathbb{H}}$. By results in [5] these must be trees (see below for the definition). For each tree $\mathbb{T}$ there exists a *dual*, i.e. a structure $D(\mathbb{T}_i)$ such that $\mathbb{G} \longrightarrow D(\mathbb{T})$ if and only if $\mathbb{T} \not\longrightarrow \mathbb{G}$, i.e. $\{\mathbb{T}\}$ is a complete set of obstructions for $D(\mathbb{T}_i)$; notice that $\mathbb{T}$ is a critical obstruction if and only if it is a core. Then it is easy to see that the product

$$D(\mathbb{T}_1) \times \cdots \times D(\mathbb{T}_s)$$

admits $\{\mathbb{T}_1, \dots, \mathbb{T}_s\}$ as a complete set of obstructions: indeed, a structure admits a homomorphism to the product if and only if it admits a homomorphism to each factor. It then follows immediately that $\widehat{\mathbb{H}}$ is the core of this product. Combining this with Lemma 3.5 we have the following:

Proposition 3.6. *The structure $\widehat{\mathbb{H}}$ admits an nuf of arity k if and only if is a retract of a product of finitely many structures $D(\mathbb{T})$ where $\mathbb{T}$ is a core tree with at most $k-1$ coloured elements.*

(I will recap the proof in detail in the proof of Theorem 3.10.)

In fact, notice that the nature of $D(\mathbb{T})$ is actually irrelevant for this last result, since it must admit a k-ary nuf by Lemma 3.5 by the simple fact that it admits $\mathbb{T}$ as a single critical obstruction. However, we will require at some point an explicit construction of some dual structure; one is given in [5], we present it here. First we define the notion of tree (see [3].)

We define the *incidence multigraph* $\mathrm{Inc}(\mathbb{T})$ of $\mathbb{T}$ as the bipartite multigraph with parts T and $\mathrm{Block}(\mathbb{T})$ which consists of all pairs (R, r) such that $R \in \sigma'$ and $r \in R(\mathbb{T})$, and with edges $e_{a,i,B}$ joining $a \in A$ to $B = (R, (x_1, \dots, x_r)) \in \mathrm{Block}(\mathbb{A})$ when $x_i = a$. Roughly speaking, one colour class consists of all vertices of $\mathbb{T}$, the other ($\mathrm{Block}(\mathbb{T})$) consists of all tuples that appear in the relation $E(\mathbb{T})$ or in some $S_h(\mathbb{T})$ (with repetitions: if an element appears in several $S_h(\mathbb{T})$ it will appear as many times in the multigraph); a vertex t is adjacent to a tuple if it appears in it.

The structure $\mathbb{T}$ is a *tree* if its associated multigraph is a tree, i.e. it is connected and acyclic.

Question. *(for Tomás ?) Is it true that if a structure has tree duality in the sense of Feder-Vardi then it has tree duality in this sense also ? (There are "less" trees as defined above, i.e. the present definition is more restrictive)*

Now we define a structure $\mathbb{D} = D(\mathbb{T})$ which is a dual to the tree $\mathbb{T}$. Let $Y = Y(\mathbb{T})$ denote the set of *hyperedges* of the tree $\mathbb{T}$, (i.e. in our setting, the union of the relations $E(\mathbb{T})$ and $S_h(\mathbb{T})$ for all $h \in H$.) The vertices of $\mathbb{D}$ are all functions $f : T \to Y$ such that $f(t)$ is incident to t; two functions f and g are *not* in the relation $E(\mathbb{D})$ if there exists some edge $xy \in E(\mathbb{T})$ such that $f(x) = g(y) = xy$; a function is *not* in $S_h(\mathbb{D})$ if there exists some $t \in S_h(\mathbb{T})$ such that $f(t) = t$.

It is in general difficult to work with this definition, so we shall spend some time (see section 5) trying to reduce the structure $D(\mathbb{T})$ to something a bit more manageable. We'll give several examples also as we go along. (See next result)

3.4. **Generators of the variety.** By Proposition 3.6 we can now get an explicit description of a family of generators for the variety of directed bipartite graphs that admit a k-ary nuf.

We start by listing a few basic critical obstructions (and their duals) that will be useful. We'll say a structure is *non-trivial* if it has at least 2 vertices. A structure is *connected* if its underlying undirected graph is connected.

Lemma 3.7. *Let $\widehat{\mathbb{H}}$ be the structure (of type σ') associated to the graph $\mathbb{H}$ (that admits an nuf). Assume $\mathbb{H}$ is connected and non-trivial. Then the following are critical obstructions of $\widehat{\mathbb{H}}$ (see the figure below):*

(1) *for all $a, b \in H$ distinct, a single vertex coloured with a and b;*
(2) *for every u in the top part of $\widehat{\mathbb{H}}$, a single directed edge, with the bottom vertex coloured by u;*
(3) *for every d in the bottom part of $\widehat{\mathbb{H}}$, a single directed edge, with the top vertex coloured by d;*
(4) *the directed path of length 2 (with no colours).*

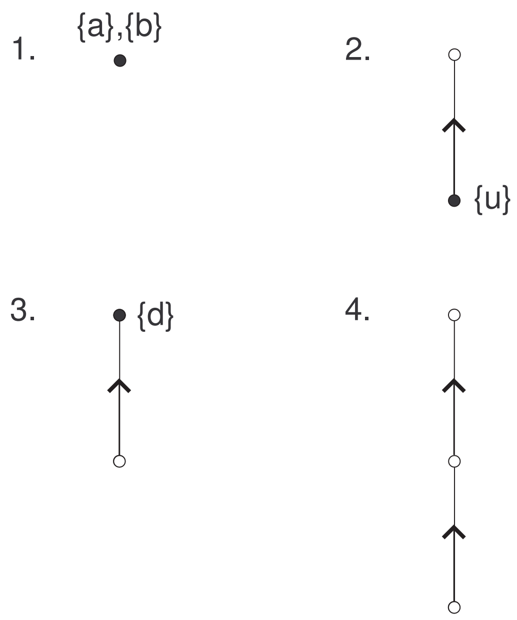

FIGURE 1. Basic critical obstructions.

Proof. This is a simple exercise. □

We illustrate the definitions we introduced above, we compute the dual $\mathbb{D} = D(\mathbb{T})$ where $\mathbb{T}$ is an obstruction in our list. Figure 2 shows the multigraphs, where the vertices of the trees are labelled, from top to bottom, (1) x, (2) and (3) x, y, and (4) x, y, z. Figure 3 shows the duals.

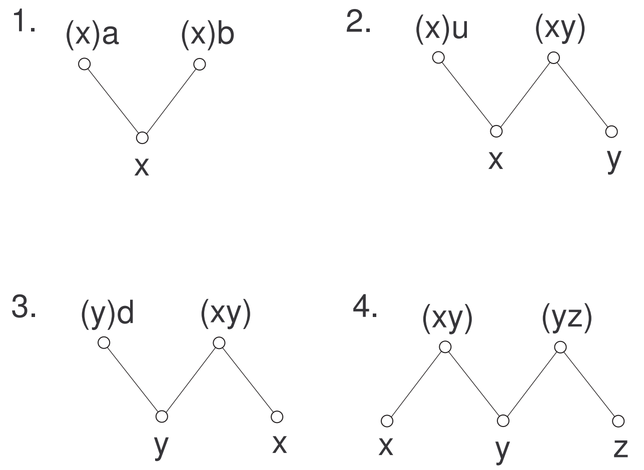

FIGURE 2. The associated multigraphs.

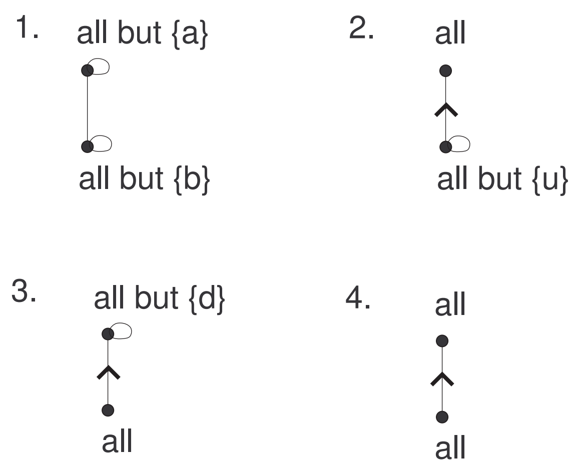

FIGURE 3. The associated duals. "All" indicates that the vertex is in every relation S_h.

Lemma 3.8. *Let* $\widehat{\mathbb{H}}$ *be as above. Then every critical obstruction* $\mathbb{T}$ *other than those mentioned in Lemma 3.7 satisfies the following conditions:*

(1) $\mathbb{T}$ *is a tree that contains no directed path of length 2, i.e. it is a bipartite (tree) digraph;*
(2) *every coloured element is coloured by a unique element, and is a leaf of* $\mathbb{T}$*;*

(3) *if an element is coloured by an element from the top (bottom) of* $\mathbb{H}$ *then it is in the top (bottom) of* $\mathbb{T}$.
(4) *no two coloured vertices are adjacent;*
(5) *every leaf is coloured;*
(6) *if a vertex is adjacent to two coloured vertices the colours on these vertices are distinct.*

Proof. Most of the above follow easily from the last lemma and the definition of critical obstruction. For the second part of (2): for this suppose that a vertex x of $\mathbb{T}$ is coloured and has (without loss of generality) out degree 2 or more; removal of the vertex x creates several subtrees of $\mathbb{T}$; to each of these we add back a copy of the vertex x with its original colour. By minimality of $\mathbb{T}$ there exist homomorphisms from each of these structures to $\widehat{\mathbb{H}}$, and since they all agree at x we can "glue" these maps back together to get a full homomorphism from $\mathbb{T}$ to $\widehat{\mathbb{H}}$, a contradiction. For (4), simply remove the edge joining the two coloured elements, etc. For (5), remove the leaf, find a homomorphism on the remaining tree, then extend it back. □

It follows from the last lemma that the obstructions we shall be considering look basically like the one depicted in Figure 4: we start with a tree which is directed bipartite (the uncoloured vertices), and coloured vertices are "glued" to it here and there, so that colours appearing on top vertices are "top" colours and same for bottom; furthermore no uncoloured vertices are left dangling, and the vertices glued to a same vertex receive distinct colours.

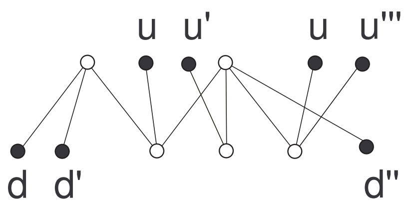

FIGURE 4. A typical "nice" obstruction. All edges are implicitly directed bottom to top.

Some temporary bad terminology: For the moment let us call trees satisfying the properties of Lemma 3.8 "nice". For the moment, we'll say that a structure $\mathbb{G}$ is "basic" if it is obtained as follows: given a nice tree $\mathbb{T}$, let $\mathbb{G} = \mathbb{P}_1 \times D(\mathbb{T})$ where $\mathbb{P}_1$ is the structure depicted in Figure 3 (4) and $D(\mathbb{T})$ is any dual of the tree $\mathbb{T}$. If the tree $\mathbb{T}$ has at most $k-1$ coloured vertices, we'll say that $\mathbb{G}$ is "$k-1$-coloured". The idea of taking the product with the path $\mathbb{P}_1$ is to obtain a structure whose underlying digraph is bipartite.

Let $\overrightarrow{\mathbb{H}}$ be a connected, non-trivial bipartite digraph, and let D (Down) and U (Up) denote its colour classes. Let $F(\mathbb{H})$ denote the structure of type σ' which is the ordinal sum of D and U, i.e. $xy \in E(F(\mathbb{H}))$ if and only if $x \in D$ and $y \in U$ and let $S_h(F(\mathbb{H})) = \{h\}$ for all $h \in H$ (see Figure 5.)

Lemma 3.9. *The structure* $F(\mathbb{H})$ *is the core dual of the set of obstructions described in Lemma 3.7.*

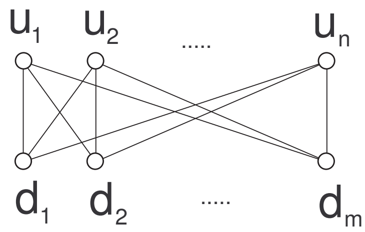

FIGURE 5. The structure $F(\mathbb{H})$. All edges are implicitly directed bottom to top. Here $D = \{d_1, \ldots, d_m\}$ and $U = \{u_1, \ldots, u_n\}$.

Proof. The fact that this structure is a core is obvious; it is also very easy to see that none of the trees described in Lemma 3.7 admits a homomorphism into it. Now suppose that $\mathbb{G}$ is a structure of type σ' such that none of these trees admits a homomorphism to $\mathbb{G}$; then it has no loops and is directed bipartite (otherwise the path of length 2 would map into it), no element is coloured by two distinct colours and no element at the bottom (top) is coloured by an element from U (D respectively). Let $d \in D$ and $u \in U$. The map $f : \mathbb{G} \to F(\mathbb{H})$ that sends a vertex to its colour if it has one, or to d if it is at the bottom or to u otherwise, is clearly a homomorphism. □

Theorem 3.10. *Let $\overrightarrow{\mathbb{H}}$ be a connected, non-trivial bipartite digraph. Then the structure $\widehat{\mathbb{H}}$ admits an nuf of arity k if and only if it is a retract of a product of finitely many $k-1$-coloured basic structures and of the structure $F(\mathbb{H})$.*

Proof. (⇐) By Lemma 3.5 any dual of a nice tree with at most $k-1$ coloured vertices admits a k-ary nuf. It is easy to verify that $F(\mathbb{H})$ and $\mathbb{P}_1$ both admit a majority operation. Consequently the product of $\mathbb{P}_1$, $F(\mathbb{H})$ and such duals also admits a k-ary nuf, hence so do retracts of these.

(⇒) Suppose that $\overrightarrow{\mathbb{H}}$ admits a k-ary nuf. Then by Lemma 3.4 the structure $\widehat{\mathbb{H}}$ has finite duality, and hence has finitely many critical (tree) obstructions $\mathbb{T}_1, \ldots, \mathbb{T}_s$; by Lemma 3.5 each of these trees has at most $k-1$ coloured elements. Now we will work in the category of structures of type σ'. Let $D(\mathbb{T}_i)$ denote any structure that admits $\mathbb{T}_i$ as unique critical obstruction. Then clearly $D(\mathbb{T}_1) \times \cdots \times D(\mathbb{T}_s)$ is a structure that admits $\{\mathbb{T}_1, \ldots, \mathbb{T}_s\}$ as its set of critical obstructions. Since we are working in type σ' it is clear that $\widehat{\mathbb{H}}$ is a core; and by definition of complete set of obstruction it is homomorphically equivalent to $D(\mathbb{T}_1) \times \cdots \times D(\mathbb{T}_s)$, and hence it is the core of this product, i.e. $\widehat{\mathbb{H}}$ is a retract of the product of the $D(\mathbb{T}_i)$.

Let $\mathbb{T}_{r+1}, \ldots, \mathbb{T}_s$ denote the nice trees in our set of obstructions; By Lemmas 3.7, 3.8 and 3.9, the structure $D(\mathbb{T}_1) \times \cdots \times D(\mathbb{T}_s)$ is homomorphically equivalent to the structure $F(\mathbb{H}) \times (D(\mathbb{T}_{r+1}) \times \cdots \times D(\mathbb{T}_s))$.

Notice that $\mathbb{P}_1 \times F(\mathbb{H})$ consists of a copy of $F(\mathbb{H})$ together with isolated vertices. It follows that

$$\begin{aligned} & F(\mathbb{H}) \times (\mathbb{P}_1 \times D(\mathbb{T}_{r+1})) \times \cdots \times (\mathbb{P}_1 \times D(\mathbb{T}_s)) \\ \simeq \quad & (F(\mathbb{H}) \times \mathbb{P}_1 \times \cdots \times \mathbb{P}_1) \times D(\mathbb{T}_{r+1}) \times \cdots \times D(\mathbb{T}_s) \\ \simeq \quad & (F(\mathbb{H}) \times D(\mathbb{T}_{r+1}) \times \cdots \times D(\mathbb{T}_s)) + \text{ isolated vertices} \end{aligned}$$

Since $\widehat{\mathbb{H}}$ is a non-trivial connected retract of $F(\mathbb{H}) \times (D(\mathbb{T}_{r+1}) \times \cdots \times D(\mathbb{T}_s))$, it follows that it is a retract of $F(\mathbb{H}) \times (\mathbb{P}_1 \times D(\mathbb{T}_{r+1})) \times \cdots \times (\mathbb{P}_1 \times D(\mathbb{T}_s))$ which completes our proof.

□

The structure $F(\mathbb{H})$ can be replaced by paths: we'll say a structure of type σ' is a *path* if its underlying digraph is a bipartite digraph which is also a path (see Figure 6.)

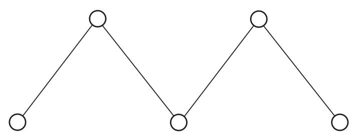

FIGURE 6. A path. Some vertices might be coloured. **All edges are implicitly directed bottom to top.**

Lemma 3.11. *The structure $F(\mathbb{H})$ is a retract of a product of paths.*

Proof. By Lemma 3.9 we have that $F(\mathbb{H})$ is a retract of the product of all the duals depicted in Figure 3. It is a simple exercise to verify that the product of $\mathbb{P}_1$ with any of the duals there is the disjoint union of a path and some isolated vertices. □

We immediately get the following:

Corollary 3.12. *Let $\overrightarrow{\mathbb{H}}$ be a connected, non-trivial bipartite digraph. Then the structure $\widehat{\mathbb{H}}$ admits an nuf of arity k if and only if it is a retract of a product of finitely many paths and $k-1$-coloured basic structures.*

□

By looking only at the underlying digraph structure we obtain:

Corollary 3.13. *Let $\overrightarrow{\mathbb{H}}$ be a connected, non-trivial bipartite digraph. Then $\overrightarrow{\mathbb{H}}$ admits an nuf of arity k if and only if it is a retract of a product of finitely many paths and (underlying digraphs of) $k-1$-coloured basic structures.*

Proof. Immediate by Lemmas 3.3 and 3.11 and Corollary 3.12. □

If at this point you are bothered by the fact that the type (signature) we are using actually *depends on the structure* $\mathbb{H}$ *we start with*, good for you ! However, magically, it does not matter in the end ! Indeed:

Remark. Concerning Theorem 3.10: it appears at first glance that the argument is circular: indeed, the description of the basic structures used to represent the (di)graph $\mathbb{H}$ depends on the type σ' which clearly depends on $\mathbb{H}$. However, closer inspection shows that this is *not* the case ! Indeed, the key point is that,if we start with *any* nice tree, it has some coloured vertices, but the definition of the dual of this nice tree is clearly independent of the “nature” of the colours; in other words, we may construct the basic structure $\mathbb{G} = \mathbb{P}_1 \times D(\mathbb{T})$ associated to the nice tree $\mathbb{T}$ *without any reference to any structure* $\mathbb{H}$ *whatsoever.*

4. Some immediate applications

4.1. **Undirected, irreflexive graphs, revisited.** We can get a similar result for undirected, irreflexive graphs as follows. Recall that if $\mathbb{H}$ is an undirected bipartite graph then it admits a k-ary nuf if and only if $\overrightarrow{\mathbb{H}}$ does (Lemma 3.3). Now Corollary 3.13 provides a criterion for the directed case. If we are given a bipartite digraph $\mathbb{G}$, let $\mathbb{G}_u$ denote the underlying undirected graph. We get the following:

Theorem 4.1. *Let* $\mathbb{H}$ *be a non-trivial, connected, undirected irreflexive graph. Then* $\mathbb{H}$ *admits a* k*-ary nuf if and only if it is a retract of a product of finitely many paths and graphs* $\mathbb{G}_u$ *where the* $\mathbb{G}$ *are* $k-1$*-coloured basic structures.*

Proof. ($\Leftarrow$) By Lemma 3.3, each of the graphs $\mathbb{G}_u$ and $F(\mathbb{H})_u$ admits a k-ary nuf and hence so do the retracts of their product.

($\Rightarrow$) By Lemma 3.2 if $\mathbb{H}$ admits an nuf then it is bipartite. By Lemma 3.3 and Corollary 3.13 the digraph $\overrightarrow{\mathbb{H}}$ is a retract of a product of paths and finitely many $k-1$-coloured basic structures. So all we need is to prove the following claim:

Claim. If the connected, bipartite digraph $\mathbb{C}$ is a retract of the product of the connected, bipartite digraphs $\mathbb{A}$ and $\mathbb{B}$ then $\mathbb{C}_u$ is a retract of $\mathbb{A}_u \times \mathbb{B}_u$.

Proof of Claim. Obviously we have that $\mathbb{C}_u$ is a retract of $(\mathbb{A} \times \mathbb{B})_u$. Now simply notice that $\mathbb{A}_u \times \mathbb{B}_u$ has two connected components, one of which is isomorphic to $(\mathbb{A} \times \mathbb{B})_u$. □

4.2. **Structures admitting a majority operation.** By Corollary 3.12 the structure $\widehat{\mathbb{H}}$ admits a majority operation if and only if it is a retract of paths and 2-coloured basic structures. Now it is clear that the only nice trees which have only 2 coloured vertices are paths with the endpoints coloured. It is not hard to determine that the basic structures associated to these are paths (see the computation in section 5). Hence we obtain immediately the following result:

Theorem 4.2. *The structure* $\widehat{\mathbb{H}}$ *admits a majority operation if and only if it is a retract of a product of paths.*

□

The undirected version of this result follows immediately, it was first proved by Bandelt et al. (get ref) ♠

4.3. **Reflexive, undirected graphs.** To do. Finite duality follows from [4], and just follow the pattern of section 3. ♠

5. Basic structures

Now we attack the problem of understanding the basic structures. Unfortunately, the dual construction is intricate, so we'd like to work with a smaller dual. We'll do this in a slightly roundabout way, by taking *retracts* of the basic structures, i.e. our claim is that *we may replace, in the statement of the main theorem, a basic structure* $\mathbb{P}_1 \times D(\mathbb{T})$ *by any of its retracts.* Obviously, if $\mathbb{G}'$ is a retract of a

$k-1$-coloured basic structure then it admits a k-ary nuf; and conversely if $\widehat{\mathbb{H}}$ is the core of a product $\mathbb{G}_1 \times \cdots \times \mathbb{G}_s$, then it is also the core of $\mathbb{G}'_1 \times \cdots \times \mathbb{G}'_s$ if for all i $\mathbb{G}'_i$ is any retract of $\mathbb{G}_i$. Hence, given a nice tree $\mathbb{T}$, we may choose any convenient retract of the structure $\mathbb{P}_1 \times D(\mathbb{T})$. For instance, if we can prove that any retract has an underlying graph which is chordal extensible, then we're done.

We now give some insight on the nature of the basic structures. We'll use some simple examples to start with. Recall the definition of the dual of a tree $\mathbb{T}$ from section 3.3: the vertices of the dual are functions from the vertex set T to the set of hyperedges (in our case, edges or colours) such that t and $f(t)$ are incident. Consider the product $\mathbb{P}_1 \times D(\mathbb{T})$: let us denote the bottom vertex of $\mathbb{P}_1$ by 0 and the top by 1. The bottom vertices of $\mathbb{P}_1 \times D(\mathbb{T})$ are of the form $(0, f)$ and the top ones are of the form $(1, g)$. By definition there is no edge from $(0, f)$ to $(1, g)$ precisely when there exists an edge xy of $\mathbb{T}$ such that $f(x) = xy = g(y)$. Notice that the value of f at vertex y plays no role, and the same for the value of g at x. This means that, given 2 bottom vertices $(0, f)$ and $(0, f')$, if f and f' agree on all bottom vertices, then *they are indistinguishable in* $\mathbb{P}_1 \times D(\mathbb{T})$ (although we must take care with the colours, see the proof below). Hence we may choose a single representative for each, retract by mapping all others to it, and similarly for every top vertex $(1, g)$. Thus we obtain the following description of a more manageable dual. Let $G(\mathbb{T})$ be the following structure. Let D and U denote the colour classes of $\mathbb{T}$. The vertices of $G(\mathbb{T})$ are split in two colour classes, the functions f from D to hyperedges of $\mathbb{T}$ such that d is incident to $f(d)$ for all $d \in D$, and the functions g from U to hyperedges of $\mathbb{T}$ such that u is incident to $g(u)$ for all $u \in U$. The edge relation and unary relations S_h are defined as usual, namely $f \to g$ unless there exists an edge xy of $\mathbb{T}$ such that $f(x) = g(y) = xy$, and $f \in S_h(G(\mathbb{T}))$ unless there exists a vertex x of $\mathbb{T}$ coloured by h such that $f(x) = h$.

Lemma 5.1. *Let* $\mathbb{T}$ *be a nice tree. Then the structure* $G(\mathbb{T})$ *described above is a retract of the structure* $\mathbb{P}_1 \times D(\mathbb{T})$.

Proof. For every $u \in U$ let $e(u)$ be some edge incident to u. For every bottom vertex $(0, f)$ in $\mathbb{P}_1 \times D(\mathbb{T})$, define another bottom vertex $(0, f')$ as follows: $f'(d) = f(d)$ for all $d \in D$, and $f(u) = e(u)$ for all $u \in U$. We claim that the map that sends $(0, f)$ to $(0, f')$ and fixes all top vertices $(1, g)$ is a structure homomorphism from $\mathbb{P}_1 \times D(\mathbb{T})$ to itself (it is clearly a retraction). Indeed, as we pointed out above, if $(0, f) \to (1, g)$ then $(0, f') \to (1, g)$ also. Hence it suffices to prove that if $(0, f) \in S_h$ then so is $(0, f')$. If h is a bottom vertex of $\mathbb{H}$ then this is clear because only bottom vertices of $\mathbb{T}$ can get colour h, and f and f' coincide on D. For $h \in U$, we defined f' so that it lies in every S_h with $h \in U$ so we're done.

Similarly, we can retract every top vertex $(1, g)$ to a dominating vertex $(1, g')$, and it is clear that the resulting structure is isomorphic to $G(\mathbb{T})$. □

Another simple remark: let y be a non-coloured vertex in the top of $\mathbb{T}$, and let $x_1, \dots, x_r$ be its neighbours in the bottom part. Then if an element f maps every x_i to the edge $x_i y$, then the element f will be isolated in $G(\mathbb{T})$ since every g in the top part must map y to some edge $x_i y$.

We illustrate the above with the simplest nice trees, namely paths with coloured endpoints (and this will complete the proof of Theorem 4.2.)

So let $\mathbb{T}$ be a path with endpoints coloured by a and b; we'll denote the vertices of the path by $\{0, 1, \dots, m\}$.

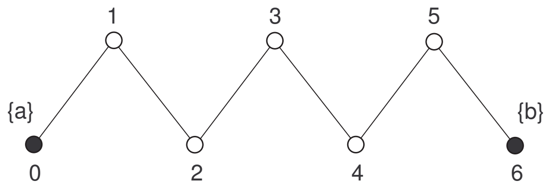

Figure 7. A path obstruction.

Let's compute the structure $G(\mathbb{T})$: let f_0 be a (non-isolated) bottom element such that $f_0(0) = 01$; then it is easy to see by the remark above that all other values of f_0 are forced to $f_0(2) = 23$, $f_0(4) = 45$, etc. The right endpoint of $\mathbb{T}$, if it is even, will be forced to $f_0(m) = m$. It is clear that f_0 has a unique neighbour, namely $f_1(1) = 12$, $f_1(3) = 34$ and so on. Now f_1 has a unique neighbour other than f_0, call it f_2, which is the same as f_0 except that $f_2(0) = 0$. Now f_2 has exactly one other neighbour, f_4, which is equal to f_2 except $f_4(1) = 01$. And then it is clear that we will obtain a path of length equal to *length of* $\mathbb{T}$ $+2$.

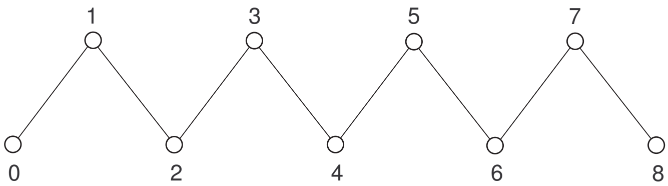

Figure 8. The structure $G(\mathbb{T})$ for the above tree obstruction. There are various colours on the vertices.

Although the above is sketchy, let's state it anyway as a result:

Lemma 5.2. *If* $\mathbb{T}$ *is a nice tree with 2 coloured vertices, then it is a path with coloured endpoints, and the structure* $G(\mathbb{T})$ *is a path.*

□

(should we formalise this ?)

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