# GRAPHS ADMITTING $k$-NU OPERATIONS. PART 1: THE REFLEXIVE CASE

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Abstract. We describe a generating set for the variety of reflexive graphs that admit a compatible $k$-ary near-unanimity operation; we further delineate a very simple subset that generates the variety of *$j$-absolute retracts*; in particular we show that the class of reflexive graphs with a 4-NU operation coincides with the class of 3-absolute retracts. Our results generalise and encompass several results on NU-graphs and absolute retracts.

## 1. Introduction

Let $k \geq 3$ be an integer. A $k$-ary operation $f$ is a *near-unanimity (NU) operation* if it satisfies, for every $1 \leq i \leq k$ the identity

$$f(x, \ldots, x, \underbrace{y}_{i}, x, \ldots, x) = x.$$

Relational structures invariant under near-unanimity operations possess remarkable properties; algebras and graphs admitting NU operations, and especially the better known case of 3-ary or *majority* operations, have been studied extensively ([1], [2], [3], [4], [5], [9], [10], [13], [21], [24], [25], [29], [33]).

The notion of NU operation is apparently due to Huhn (see [1]) and arises naturally in the classification of 2-element algebras (see [30]). Baker and Pixley were the first to give several conditions on equational classes of algebras equivalent to the existence of a near-unanimity term [1]. Equational classes with an NU term are known to be congruence-distributive, a very well-behaved and widely studied family of varieties of algebras; it has recently been proved that for algebras stemming from structures of finite signature, congruence-distributivity is actually equivalent to the presence of an NU term [3]. NU operations have also been studied in the context of algebraic dualities (see for instance [9]). Furthermore structures admitting compatible NU operations (a.k.a. *NU polymorphisms*) are well-behaved from a computational complexity point of view: in their seminal 1993 paper, Feder and Vardi proved that constraint satisfaction problems (CSPs) whose constraints are invariant under an NU operation can be solved in polynomial time via the query

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language Datalog; we now know that in fact these CSPs are describable in Linear Datalog, and in particular are solvable in nondeterministic Logspace [4].

Parallel to the study of NU operations on algebras, the concept of *absolute retract* was being developed in the framework of graphs. The graph $H$ is an absolute retract (with respect to isometry) if every isometric embedding of $H$ is a coretraction, where the metric is the usual graph distance. P. Hell showed that in the category of bipartite graphs with edge-preserving maps, the absolute retracts are precisely the retracts of products of paths [14]. An analogous result was proved for reflexive, undirected graphs in [19] by viewing graphs as finite metric spaces over a Heyting algebra (see also [28, 18]). In the same paper, Jawhari, Misane and Pouzet show that the reflexive (di)graphs that are absolute retracts are precisely those admitting a compatible majority operation:

**Theorem 1.1.** [19] *Let $\mathbb{H}$ be a reflexive graph. Then the following conditions are equivalent:*

(1) $\mathbb{H}$ *admits a compatible majority operation;*
(2) $\mathbb{H}$ *is an absolute retract;*
(3) $\mathbb{H}$ *is a retract of a product of paths.*

The first result of this form connecting NU operations and absolute retracts is attributed to I. Rival who had shown that posets admitting a majority operation are the retracts of products of fences (see [19, 29]). The analog for bipartite, undirected graphs is due to Bandelt [2]. It should be noted that for general reflexive digraphs, the class of absolute retracts is strictly larger than that of retracts of products of paths (but see [20] for an analogous description.)

Although some work was done in trying to generalise the classification results for majority strutures to those admitting NU operation of higher arity (see for instance [5], [10], [29], [31] and [32]), for a while the problem seemed quite hopeless. One major obstacle is that the metric point of view used to prove the results in the majority case seems impossible to adapt for arities 4 and up, and a new approach seemed necessary. The existence of a finite duality, for any core relational structure, implies the existence of an NU polymorphism [21], but the converse is not true in general. The converse does hold for reflexive graphs, however, and this yields the new approach that was needed. In the present paper we exploit the finite duality of NU graphs to provide a family of generating graphs using tree duals [27]. The converse also holds for strongly bipartite digraphs, and in [12], the sequel to this paper, we use a similar approach to provide families that generate the classes of strongly bipartite digraphs, and simple graphs, that admit NU polymorphisms.

The notion of absolute retract with respect to isometry has been generalised to the notion of absolute retract with respect to *holes* (see [6], [14], [18] and [23]). Roughly speaking, a $j$-hole in a graph is an empty intersection of $j$ disks which is minimal with respect to this property. A graph is a $j$-absolute retract if it is a retract of any graph in which it embeds in such a way that its $j'$-holes are not filled, for all $j' \leq j$. It is known that a $j$-absolute retract admits a $(j+1)$-ary NU polymorphism [23] and that the converse holds for $j = 2$. For $j > 3$ there are graphs admitting a $j+1$-ary NU operation that are not absolute retracts, but it has been a long-standing open problem to determine the case $j = 3$: we settle this here by showing that 3-absolute retracts are precisely the graphs admitting a 4-NU polymorphism (Theorem 5.17).

We now outline briefly the contents of the paper. We consider only reflexive, undirected graphs; the sequel [12] gathers our results in the case of strongly bipartite graphs and simple graphs. Section 2 contains preliminary results and definitions on relational structures, graphs and polymorphisms. Section 3 describes the connection between NU operations and finite duality: the main result of this section, Theorem 3.9, shows that every graph admitting a $k$-ary NU polymorphism is a retract of a product of reflexive duals of trees with at most $k-1$ leaves. In Section 4 we describe the building blocks explicitly in two different ways, as subgraphs of hypercubes (Corollaries 4.4 and 4.8.) In Section 5 we prove that absolute retracts are precisely the graphs admitting *polyad* duality, where polyads are subdivisions of stars, i.e. trees obtained by glueing several paths at a unique central vertex (Theorem 5.7). We also show that duals of non-polyad trees are not absolute retracts, and that the class of reflexive graphs with a 4-NU operation coincides with the class of 3-absolute retracts. (Theorem 5.17).

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## 2. Preliminaries

2.1. **Structures and homomorphisms.** We refer the reader to [8] for basic notation and terminology. In the present paper we will use blackboard fonts such as $\mathbb{G}$, $\mathbb{H}$, etc. to denote relational structures and their latin equivalent $G$, $H$, etc. to denote their respective universes. A *signature* $\tau$ is a (finite) set of relation symbols with associated arities. We say that $\mathbb{H} = \langle H; R(\mathbb{H})(R \in \tau)\rangle$ is a *relational structure of signature* $\tau$ if $R(\mathbb{H})$ is a relation on $H$ of the corresponding arity, for each relation symbol $R \in \tau$; the relations $R(\mathbb{H})$ are called the *basic* or *fundamental* relations of the structure.

Let $\mathbb{G}$ and $\mathbb{H}$ be structures of signature $\tau$. A *homomorphism* from $\mathbb{G}$ to $\mathbb{H}$ is a map $f$ from $G$ to $H$ such that $f(R(\mathbb{G})) \subseteq R(\mathbb{H})$ for each $R \in \tau$. We write $\mathbb{G} \to \mathbb{H}$ to indicate there exists a homomorphism from $\mathbb{G}$ to $\mathbb{H}$. A homomorphism $r : \mathbb{H} \to \mathbb{R}$ is a *retraction* if there exists a homomorphism (called a *coretraction*) $e : \mathbb{R} \to \mathbb{H}$ such that $r \circ e$ is the identity on $\mathbb{R}$ and we say that $\mathbb{R}$ is a *retract* of $\mathbb{H}$ and write $\mathbb{R} \trianglelefteq \mathbb{H}$. A structure $\mathbb{H}$ is called a *core* if every homomorphism from $\mathbb{H}$ to itself is a permutation on $H$; note that a retract of $\mathbb{H}$ of minimum cardinality is a core, and is unique up to isomorphism [17], and hence we may speak of *the core* of the structure $\mathbb{H}$. We denote by CSP($\mathbb{H}$) the class of all $\tau$-structures $\mathbb{G}$ that admit a homomorphism to $\mathbb{H}$.

Throughout this paper we consider the usual product of $\tau$-structures, namely if $\mathbb{G}$ and $\mathbb{H}$ are $\tau$-structures then their *product* is the $\tau$-structure $\mathbb{G} \times \mathbb{H}$ with universe $G \times H$ such that, for every $R \in \tau$ of arity $r$, $((g_1, h_1), \cdots, (g_r, h_r)) \in R(\mathbb{G} \times \mathbb{H})$ if and only if $((g_1, \ldots, g_r), (h_1, \ldots, h_r)) \in R(\mathbb{G}) \times R(\mathbb{H})$. We shall consider notations such as $\prod_{i=1}^{n} \mathbb{G}_i$ and $\mathbb{H}^k$ to be self-evident.

Let $\mathbb{H}$ be a $\tau$-structure. The *retraction problem for* $\mathbb{H}$ is the following: given a structure $\mathbb{G}$ containing a copy of $\mathbb{H}$, decide if $\mathbb{G}$ retracts to $\mathbb{H}$. It is in fact equivalent under positive first-order reductions to the *one-or-all list-homomorphism* problem for $\mathbb{H}$ (see [11], and the paragraph preceding Lemma 5.6 below): an input consists of a $\tau$-structure $\mathbb{G}$ with certain vertices coloured by a pre-assigned value from $H$, and the problem is to determine if there exists a homomorphism from $\mathbb{G}$ to $\mathbb{H}$ that extends these values. For brevity's sake we shall still refer to the latter as the retraction problem. Formally, we "add constants" to structures, i.e. we add, as basic unary relations to a given structure, each of its one-element sets:

**Definition 2.1.** *Let $\mathbb{H}$ be a $\tau$-structure. For each $h \in H$, let $S_h$ be a unary relation symbol. Let $\tau_{\mathbb{H}} = \tau \cup \{S_h : h \in H\}$, and let $\mathbb{H}^c$ denote the $\tau_{\mathbb{H}}$-structure obtained from $\mathbb{H}$ by adding all relations $S_h(\mathbb{H}^c) = \{h\}$. The problem $CSP(\mathbb{H}^c)$ is called the* retraction problem for $\mathbb{H}$. *Let $\mathbb{G}$ be a $\tau_{\mathbb{H}}$-structure. We say that a vertex $x \in G$ is* coloured *if it belongs to some unary relation $S_h(\mathbb{G})$ and refer to $h$ as its colour (a vertex may have several colours). Let $\mathbb{G}^\tau$ denote the (reduct) $\tau$-structure obtained from $\mathbb{G}$ by simply removing the relations indexed by the $S_h$.*

Let $H$ be a non-empty set, let $\theta$ be an $m$-ary relation on $H$ and let $f : H^k \to H$ be a $k$-ary operation on $H$. We say that $f$ *preserves* $\theta$ if the following holds: if the $m \times k$ matrix $M$ has each column in $\theta$, then applying $f$ to the rows of $M$ yields a tuple of $\theta$. If $\mathbb{H}$ is a $\tau$-structure, and $f$ preserves each of its basic relations (equivalently, if $f$ is a homomorphism from $\mathbb{H}^k$ to $\mathbb{H}$), we say that $f$ is *compatible* with $\mathbb{H}$, or that $\mathbb{H}$ *admits* $f$; one also says that $f$ is a *polymorphism* of $\mathbb{H}$, see [8] and [30] for instance. Recall that $f$ is a $k$-ary near-unanimity ($k$-NU) operation if it satisfies, for every $1 \leq i \leq k$ the identity

$$f(x, \ldots, x, \underbrace{y}_{i}, x, \ldots, x) = x.$$

A structure is said to be *$k$-NU* if it admits a $k$-NU operation.

2.2. **Graphs and digraphs.** A *digraph* is a relational structure equipped with a single, binary relation. A *graph* $\mathbb{H}$ is a relational structure $\mathbb{H} = \langle H, E(\mathbb{H})\rangle$ where $E(\mathbb{H})$ is a binary relation which is *symmetric*, i.e. $(x, y) \in E(\mathbb{H})$ if and only if $(y, x) \in E(\mathbb{H})$. The graph $\mathbb{H}$ is *reflexive (irreflexive)* if $(x, x) \in E(\mathbb{H})$ ($(x, x) \notin E(\mathbb{H})$) for all $x \in H$.

Let $\mathbb{G}$ be any digraph; then we let $\mathbb{G}^u$ denote the underlying undirected graph, i.e. $E(\mathbb{G}^u)$ consists of all pairs $(x, y)$ such that $(x, y)$ or $(y, x)$ is in $E(\mathbb{G})$. Let $\mathbb{H}$ be a digraph. We say $\mathbb{H}$ is connected if the graph $\mathbb{H}^u$ is connected; if $\mathbb{H}$ is not connected, a set $G$ of vertices of $\mathbb{H}$ is a *connected component* of $\mathbb{H}$ if it is a connected component of $\mathbb{H}^u$.

The next result allows us to consider only connected graphs in the sequel.

**Proposition 2.2** ([5], Prop 2.1)**.** *Let $\mathbb{H}$ be a reflexive graph. Then $\mathbb{H}$ admits an NU operation of arity $k$ if and only if each of its subgraphs induced by a connected component does.*

## 3. NU operations and Finite Duality

Let $\mathbb{H}$ be a $\tau$-structure. We say that CSP($\mathbb{H}$) has *finite duality* if there exist finitely many $\tau$-structures $\mathbb{T}_1, \ldots, \mathbb{T}_s$ such that the following holds: for every $\tau$-structure $\mathbb{G}$, there is *no* homomorphism from $\mathbb{G}$ to $\mathbb{H}$ precisely if there is some $\mathbb{T}_i$ that admits a homomorphism to $\mathbb{G}$. The set $\{\mathbb{T}_1, \ldots, \mathbb{T}_s\}$ is called a *duality* for $\mathbb{H}$.

Constraint satisfaction problems that have finite duality are arguably the simplest from a complexity-theoretic point of view, and possess many intriguing properties [21].

The key connection we need is:

**Theorem 3.1** ([22])**.** *Let $\mathbb{H}$ be a connected reflexive graph. Then the following are equivalent:*

1. (1) $\mathbb{H}$ *admits an NU operation;*
2. (2) $\mathrm{CSP}(\mathbb{H}^c)$ *has finite duality.*

3.1. **Trees and Duals.** We shall require the notion of a $\tau$-tree (see [21].) Let $\mathbb{T}$ be a $\tau$-structure. We define the *incidence multigraph* $\mathrm{Inc}(\mathbb{T})$ of $\mathbb{T}$ as the bipartite multigraph with parts $T$ and $\mathrm{Block}(\mathbb{T})$, where the latter consists of all pairs $(R, r)$ such that $R \in \tau$ and $r \in R(\mathbb{T})$, and with edges $e_{a,i,B}$ joining $a \in T$ to $B = (R,(x_1,\dots,x_r)) \in \mathrm{Block}(\mathbb{T})$ when $x_i = a$. Roughly speaking, one colour class consists of all vertices of $\mathbb{T}$, the other ($\mathrm{Block}(\mathbb{T})$) consists of all tuples that appear in the relations $R(\mathbb{T})$ (with repetitions: if a tuple appears in several relations it will appear as many times in the multigraph); a vertex $t$ is adjacent to a tuple if it appears in it, with an edge for each appearance. The structure $\mathbb{T}$ is a *$\tau$-tree* if its associated multigraph is a tree, i.e. it is connected and acyclic.

**Theorem 3.2** ([26])**.** *Let $\mathbb{H}$ be a $\tau$-structure.*

1. (1) *If $\mathbb{H}$ has finite duality then there exists a duality for $\mathbb{H}$ consisting of finitely many $\tau$-trees;*
2. (2) *(Existence of duals) Let $\mathcal{T}$ be a finite family of $\tau$-trees. Then there exists a $\tau$-structure $D$ such that $\mathcal{T}$ is a duality for $D$.*

We shall now give an explicit description, in the case of reflexive graphs, of the duals whose existence is guaranteed by Theorem 3.2 (2). For the remainder of this section and unless otherwise mentioned, the signature $\tau$ will be that of digraphs, i.e. it consists of a single binary relation symbol. Notice that in this case, if a $\tau$-structure is a $\tau$-tree then 1- and 2-cycles are forbidden, i.e. it has no loops nor any symmetric edges.

**Definition 3.3.** *Let $\mathbb{H}$ be a $\tau$-structure and let $\mathbb{T}$ be a $\tau_{\mathbb{H}}$-tree. The $\tau_{\mathbb{H}}$-structure $D(\mathbb{T})$ has universe*

$$\{f : T \to \mathrm{Block}(\mathbb{T}) : [t, f(t)] \in E(\mathrm{Inc}(\mathbb{T})) \text{ for all } t \in T\},$$

*and $(f, g)$ is* not *an edge of $D(\mathbb{T})$ if there exists an edge $e = (s,t)$ of $\mathbb{T}$ such that $f(s) = e = g(t)$; the map $f$ does not belong to $S_h(D(\mathbb{T}))$ if there exists some $t \in S_h$ such that $f(t) = (t)$.*

**Theorem 3.4** ([27])**.** *Let $\mathbb{H}$ be a $\tau$-structure and let $\mathbb{T}$ be a $\tau_{\mathbb{H}}$-tree. Then $\{\mathbb{T}\}$ is a duality for $D(\mathbb{T})$. if $\mathbb{T}_1,\dots,\mathbb{T}_s$ are $\tau_{\mathbb{H}}$-trees, then $\{\mathbb{T}_1,\dots,\mathbb{T}_s\}$ is a duality for $\prod_{i=1}^s D(\mathbb{T}_i)$.*

3.2. **Reflexive Duals.** If $\mathbb{D}$ is a digraph with at least one loop, we let $\mathbb{D}_r$ denote the reflexive, undirected subgraph of $\mathbb{D}$ whose vertices are the loops of $\mathbb{D}$, and whose edges are the pairs $(u, v)$ such that both $(u, v)$ and $(v, u)$ are arcs of $\mathbb{D}$.

**Lemma 3.5.** *Let $\mathbb{G}, \mathbb{H}$ be digraphs each containing at least one loop.*

1. (1) $(\mathbb{G} \times \mathbb{H})_r = \mathbb{G}_r \times \mathbb{H}_r$*;*
2. (2) *If $\mathbb{G} \trianglelefteq \mathbb{H}$ then $\mathbb{G}_r \trianglelefteq \mathbb{H}_r$.*

*Proof.* (1) is straightforward. For (2) notice that any edge-preserving map will take loops into loops and symmetric edges into symmetric edges, hence the retraction and coretraction on $\mathbb{G}$ and $\mathbb{H}$ restrict to the desired maps on $\mathbb{G}_r$ and $\mathbb{H}_r$. □

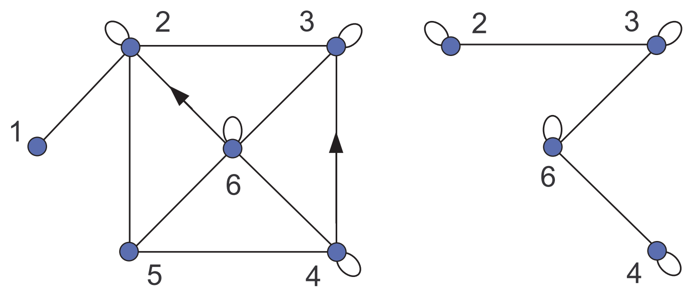


FIGURE 1. A digraph $\mathbb{D}$ and its graph $\mathbb{D}_r$.

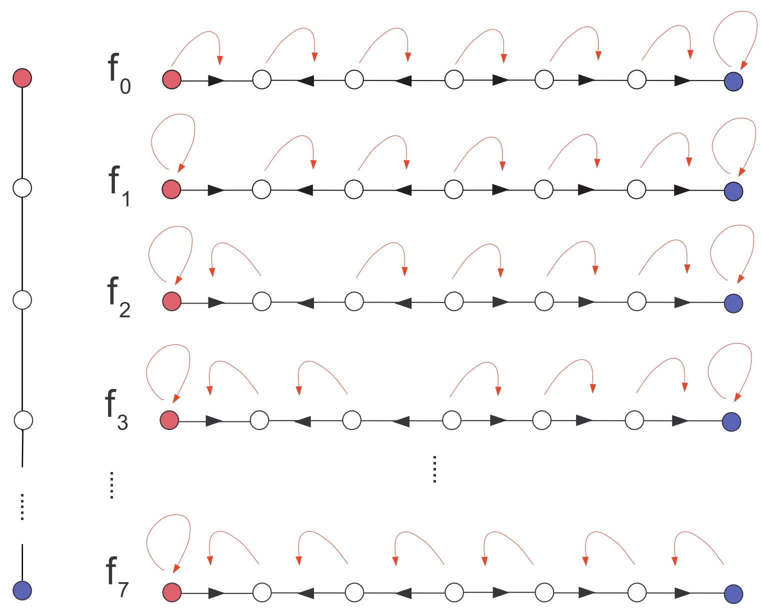


FIGURE 2. The dual $D_r(\mathbb{T})$ for a path $\mathbb{T}$ of length 6, and the representation of its vertices as functions $f_i$ on the vertices of $\mathbb{T}$. A red arrow indicates the value of $f_i$ on a given vertex.

Let $\mathbb{H}$ be a reflexive graph, and let $\mathbb{T}$ be a $\tau_{\mathbb{H}}$-tree. Notice that $(\mathbb{T}^{\tau})^u$ is a tree. We say $\mathbb{T}$ is *elementary* if either (a) it consists of a single vertex lying in two distinct unary relations $S_h$ and $S_{h'}$, or (b) its coloured elements are precisely the leaves of $(\mathbb{T}^{\tau})^u$, each lying in exactly one unary relation $S_h$.

Notice that if $\mathbb{T}$ is an elementary $\tau_{\mathbb{H}}$-tree then $D(\mathbb{T})$ contains at least one loop: indeed, if $\mathbb{T}$ has a single vertex lying in two distinct unary relations, then clearly its dual contains two loops. Now suppose that $\mathbb{T}$ has at least two vertices. Choose any non-leaf vertex of $\mathbb{T}$ and view it as a root, i.e. order the tree bottom up starting at this particular vertex. Clearly we may define a map $f$ that assigns to any non-leaf vertex $x$ an edge $f(x)$ leading to a neighbour higher up in the ordering, and for any leaf $x$ define $f(x)$ to be the unary relation that contains $x$. It is immediate that $f$

is a loop. It follows that $D(\mathbb{T})_r$ is well-defined; for convenience, we will denote it by $D_r(\mathbb{T})$. Graphs of this form we call *reflexive duals.*

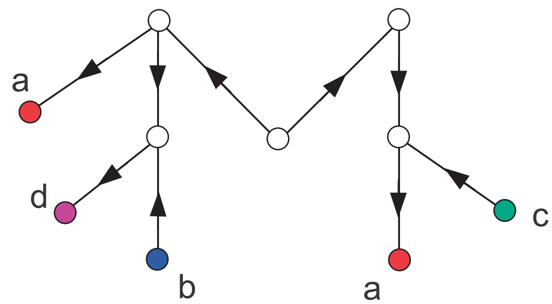


FIGURE 3. An elementary tree.

**Theorem 3.6.** *Let $\mathbb{H}$ and $\mathbb{K}$ be two reflexive graphs, let $\mathbb{S}$ be an elementary $\tau_{\mathbb{H}}$-tree, and let $\mathbb{T}$ be an elementary $\tau_{\mathbb{K}}$-tree such that $(\mathbb{S}^{\tau})^u$ and $(\mathbb{T}^{\tau})^u$ are isomorphic. Then $D_r(\mathbb{S})$ and $D_r(\mathbb{T})$ are isomorphic reflexive graphs.*

*Proof.* Since $\mathbb{S}$ and $\mathbb{T}$ are elementary and $(\mathbb{S}^{\tau})^u$ and $(\mathbb{T}^{\tau})^u$ are isomorphic, it is clear that their incidence multigraphs are isomorphic, and this induces a one-to-one correspondence between the universes of the duals. Since the edge structure of the reflexive duals does not depend on the unary relations on the trees, it is clear that this correspondence preserves edges (and in particular loops) and hence the graphs $D_r(\mathbb{S})$ and $D_r(\mathbb{T})$ are isomorphic. □

The last result states that the reflexive dual of an elementary $\tau_{\mathbb{H}}$-tree $\mathbb{S}$ is determined completely by the underlying graph tree $(\mathbb{S}^{\tau})^u$. We may therefore define, without ambiguity, the reflexive dual $D_r(\mathbb{T})$ of a (graph) tree $\mathbb{T}$ to be $D_r(\mathbb{S})$ where $\mathbb{S}$ is any elementary $\tau_{\mathbb{H}}$-tree with $(\mathbb{S}^{\tau})^u = \mathbb{T}$ for any $\tau$-structure $\mathbb{H}$. Let $f$ be an element of the reflexive dual $D_r(\mathbb{T})$, i.e. a loop in the dual. By definition of the dual and of elementary $\tau$-tree, it is immediate that $f$ may be viewed as a map from $T$ to $E(\mathbb{T}) \cup T$ such that

(1) for every $t \in T$, $f(t)$ is either an edge of $\mathbb{T}$ incident with $t$ or $f(t) = t$ (if $t$ is a leaf),
(2) there is no edge $e = (s, t)$ such that $f(s) = e = f(t)$.

3.3. **Colours, Leaves and Arity of NU polymorphisms.** Let $\tau$ be any signature and let $\mathbb{G}$ and $\mathbb{H}$ be $\tau$-structures. We say that $\mathbb{G}$ is a *substructure* of $\mathbb{H}$ if $G \subseteq H$ and $R(\mathbb{G}) \subseteq R(\mathbb{H})$ for all $R \in \tau$; if furthermore $\mathbb{G} \neq \mathbb{H}$ we say $\mathbb{G}$ is a *proper* substructure of $\mathbb{H}$. Let $\mathbb{U}$ and $\mathbb{H}$ be $\tau$-structures. We say that $\mathbb{U}$ is an *obstruction* of $\mathbb{H}$ if there is no homomorphism from $\mathbb{U}$ to $\mathbb{H}$; furthermore if every proper substructure of $\mathbb{U}$ admits a homomorphism to $\mathbb{H}$ we say $\mathbb{U}$ is a *critical* obstruction of $\mathbb{H}$. The following useful characterisation of *NU* structures is a slight adaptation of a result of Zádori [32] (see also [13]).

**Lemma 3.7.** *Let $\mathbb{H}$ be a $\tau$-structure. Then the following are equivalent:*

(1) *$\mathbb{H}$ is a $k$-NU structure;*

(2) $\mathbb{H}^c$ *is a k-NU structure;*
(3) *every critical obstruction of* $\mathbb{H}^c$ *has at most* $k-1$ *coloured elements.*

*Proof.* (1) $\Leftrightarrow$ (2): immediate since NU operations are idempotent, i.e. they satisfy $f(x,\dots,x)=x$ for all $x$, and hence preserve every one-element unary relation.

(2) $\Rightarrow$ (3): suppose that $\mathbb{H}^c$ admits an NU operation $f$ of arity $k$; then it admits NU operations of every arity larger than $k$ (just add fictitious variables). Hence it will suffice to prove that $\mathbb{H}^c$ has no critical obstruction with $k$ coloured elements. If $\mathbb{U}$ were such an obstruction, with vertices $t_i \in S_{h_i}(\mathbb{U})$ for $1 \le i \le k$, let $\mathbb{U}_i$ be the substructure obtained from $\mathbb{U}$ by removing the vertex $t_i$ from the relation $S_{h_i}(\mathbb{U})$. Since $\mathbb{U}$ is critical we have homomorphisms $f_i : \mathbb{U}_i \longrightarrow \mathbb{H}^c$ for all $1 \le i \le k$. Define a map $\phi : U \longrightarrow H$ by $\phi(x) = f(f_1(x),\dots,f_k(x))$. Obviously $f$ preserves all relations, including the $S_h$ since if $x = t_i$, then $f_j(x) = h_i$ for all $j \neq i$ and since $f$ is an NU operation it follows that $\phi(t_i) = h_i$.

(3) $\Rightarrow$ (2): suppose that every critical obstruction of $\mathbb{H}^c$ has at most $k-1$ coloured elements. Consider the $\tau_{\mathbb{H}}$-structure $\mathbb{G}$ where $\mathbb{G}^\tau = (\mathbb{H}^c)^k$ and for each $h \in H$, the unary relation $S_h(\mathbb{G})$ consists of all tuples $(x_1,\dots,x_k)$ where at least $k-1$ of the entries are equal to $h$. Obviously there exists a homomorphism from $\mathbb{G}$ to $\mathbb{H}^c$ if and only if $\mathbb{H}^c$ admits a $k$-ary NU; if it does not then there exists a critical obstruction $\mathbb{U}$ admitting a homomorphism to $\mathbb{G}$, and by hypothesis it has only $k-1$ coloured vertices. But if $x^i \in S_{h_i}$ for $1 \le i \le k-1$ are any $k-1$ coloured vertices in $\mathbb{G}$ then there exists a coordinate $j$ such that the $j$-th coordinate of $x^i$ is equal to $h_i$ for every $1 \le i \le k-1$; hence the $j$-th projection would be a homomorphism from $\mathbb{U}$ to $\mathbb{H}^c$, a contradiction. □

The next result states that the reflexive dual of a tree with $k-1$ leaves admits a $k$-ary NU operation: we defer the details of its proof to section 4 where we give an explicit description of such an operation.

**Lemma 3.8.** *Let* $k \ge 3$ *and let* $\mathbb{T}$ *be a tree with* $k-1$ *leaves. Then* $D_r(\mathbb{T})$ *is k-NU.*

*Proof.* Immediate by Lemma 4.2 (2) and Lemma 4.3, once we notice the following easy fact: if $\mathbb{R}$ is a retract of $\mathbb{G}$ and $\mathbb{G}$ is $k$-NU then $\mathbb{R}$ is $k$-NU. Indeed, if $f$ is a $k$-NU operation on $\mathbb{G}$ and $r$ is a retraction of $\mathbb{G}$ onto $\mathbb{R}$ then the restriction of $r \circ f$ to $R^k$ satisfies the desired properties. □

### 3.4. Representation of $k$-NU Graphs by Reflexive Duals.

**Theorem 3.9.** *Let* $k \ge 3$ *and let* $\mathbb{H}$ *be a connected reflexive graph. Then the following conditions are equivalent:*

(1) $\mathbb{H}$ *is k-NU;*
(2) *there exist trees* $\mathbb{T}_1,\dots,\mathbb{T}_s$*, each with at most* $k-1$ *leaves, such that*

$$\mathbb{H} \trianglelefteq \prod_{i=1}^{s} D_r(\mathbb{T}_i).$$

*Proof.* (2) $\Rightarrow$ (1): if $\mathbb{T}_i$ is a tree with at most $k-1$ leaves, then by Lemma 3.8 $D(\mathbb{T}_i)$ admits a $k$-NU polymorphism. Since each of its factors is $k$-NU the product of the $D(\mathbb{T}_i)$ is $k$-NU: just define the operation coordinatewise in the obvious fashion. Finally, as $\mathbb{H}$ is a retract of a $k$-NU graph, it is itself $k$-NU as noted above in the proof of Lemma 3.8.

(1) $\Rightarrow$ (2): if $\mathbb{H}$ is $k$-NU then $CSP(\mathbb{H}^c)$ has finite duality by Theorem 3.1, and hence $\mathbb{H}^c$ has a duality $\{\mathbb{T}_1, \ldots, \mathbb{T}_s\}$ that consists of finitely many $\tau_{\mathbb{H}}$-trees by Theorem 3.2. Notice that we may assume that each tree is in fact a critical obstruction: indeed, for each $i$ we may find an induced substructure $\mathbb{T}_i'$ of $\mathbb{T}_i$ which is minimal with the property that $\mathbb{T}_i' \not\to \mathbb{H}^c$; in particular each $\mathbb{T}_i'$ is a critical obstruction. It is also easy to see that each $\mathbb{T}_i'$ is a $\tau_{\mathbb{H}}$-tree: since $\mathrm{Inc}(\mathbb{T}_i')$ is contained in $\mathrm{Inc}(\mathbb{T}_i)$ it is acyclic, and it must also be connected because of the minimality condition. It is immediate that $\{\mathbb{T}_1', \ldots, \mathbb{T}_s'\}$ is also a finite duality for $CSP(\mathbb{H}^c)$. So we now assume that each $\mathbb{T}_i$ is critical. By Lemma 3.7 each $\mathbb{T}_i$ has at most $k-1$ coloured elements. By Theorem 3.4 we have that

$$\mathbb{H}^c \to \prod_{i=1}^{s} D(\mathbb{T}_i) \text{ and } \prod_{i=1}^{s} D(\mathbb{T}_i) \to \mathbb{H}^c.$$

Since $\mathbb{H}^c$ has every one-element subset as a basic relation, it is trivially a core, and hence it is the core of the product of the $D(\mathbb{T}_i)$, and in particular a retract of this product. Thus

$$\mathbb{H}^c \trianglelefteq \prod_{i=1}^{s} D(\mathbb{T}_i)$$

from which we get

$$\mathbb{H} \trianglelefteq \left[\prod_{i=1}^{s} D(\mathbb{T}_i)\right]^{\tau}.$$

By Lemma 3.5 and the obvious fact that taking reducts commutes with the product, we conclude that

$$\mathbb{H} = \mathbb{H}_r \trianglelefteq \left(\left[\prod_{i=1}^{s} D(\mathbb{T}_i)\right]^{\tau}\right)_r = \prod_{i=1}^{s} D_r(\mathbb{T}_i).$$

The last thing to show is that the $\tau_{\mathbb{H}}$-trees $\mathbb{T}_i$ are elementary. Let $\mathbb{T}$ be some $\mathbb{T}_i$. Recall that $(\mathbb{T}^{\tau})^u$ is a tree. Suppose that $\mathbb{T}$ has at least two vertices; since it is a critical obstruction it is obvious that any vertex lying in some $S_h$ can only belong to one of them. Next, suppose that $(\mathbb{T}^{\tau})^u$ has a leaf which is not coloured. Remove this vertex from $\mathbb{T}$ to obtain $\mathbb{T}'$. Since $\mathbb{T}$ is a critical obstruction there exists some homomorphism from $\mathbb{T}'$ to $\mathbb{H}^c$, and since $\mathbb{H}$ is reflexive we can clearly extend this homomorphism to $\mathbb{T}$, a contradiction. Finally suppose that $\mathbb{T}$ has some vertex $x$ lying in $S_h$ but is not a leaf of $(\mathbb{T}^{\tau})^u$. Choose an edge $e$ incident with $x$, and consider the substructure $\mathbb{S}$ of $\mathbb{T}$ which is induced by all vertices reachable from $x$ via $e$ (including $x$), and let $\mathbb{U}$ denote the substructure induced by all other vertices together with $x$. Since $\mathbb{T}$ is a critical obstruction there exist homomorphisms from $\mathbb{S}$ and $\mathbb{U}$ to $\mathbb{H}^c$; since both must have value $h$ on $x$, their union defines a homomorphism from $\mathbb{T}$ to $\mathbb{H}^c$, a contradiction. Thus the coloured vertices of $\mathbb{T}$ are precisely the leaves of $(\mathbb{T}^{\tau})^u$ and they each receive exactly one colour.

□

## 4. Coordinatisations

We describe two different coordinatisations of reflexive duals.

4.1. **Coordinatisation in the 3-hypercube.** We describe a simple coordinatisation. Downside: the case of paths and polyads is not transparent.

**Definition 4.1.** *Let $\mathbb{T}$ be a tree with colour classes $U$ and $D$ and edges $e_1, \ldots, e_m$. Define a graph $\mathbb{K}(\mathbb{T})$ as follows: its vertices are the tuples $(x_1, \ldots, x_m)$ such that*

(1) *$x_i \in \{0, 1, 2\}$ for every $1 \leq i \leq m$;*
(2) *for each $u \in U$ of degree greater than 1, $x_i = 2$ for exactly one edge $e_i$ incident with $u$;*
(3) *for each $d \in D$ of degree greater than 1, $x_i = 0$ for exactly one edge $e_i$ incident with $d$.*

*Tuples $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are adjacent if $|x_i - y_i| \leq 1$ for all $i$.*

*The graph $\mathbb{K}^0(\mathbb{T})$ in defined in the same manner, except we replace "exactly one" by "at least one" in conditions (2) and (3) above.*

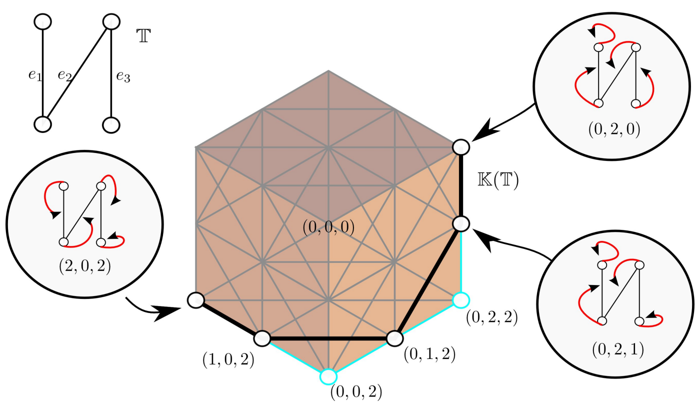


FIGURE 4. A tree $\mathbb{T}$, the graph $\mathbb{K}(\mathbb{T})$ and $\mathbb{K}^0(\mathbb{T})$ ($\mathbb{K}(\mathbb{T})$ plus the blue vertices)

**Lemma 4.2.** *Let $\mathbb{T}$ be a tree with $k-1$ leaves and $m$ edges.*

(1) *$\mathbb{K}(\mathbb{T}) \trianglelefteq \mathbb{K}^0(\mathbb{T})$;*
(2) *the graph $\mathbb{K}^0(\mathbb{T})$ admits a $k$-ary NU polymorphism.*

*Proof.* Let $\mathbb{K} = \mathbb{K}(\mathbb{T})$ and $\mathbb{K}^0 = \mathbb{K}^0(\mathbb{T})$; let $U$ and $D$ denote the colour classes of $\mathbb{T}$.

(1) If $(x_1, \ldots, x_m)$ is a vertex of $\mathbb{K}^0$ let $r(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$ be defined as follows: if $x_i = 0$, and $e_i$ is incident with a vertex $d \in D$ such that there exists an edge $e_j$ incident with $d$ with $j < i$ and $x_j = 0$, or if $x_i = 2$, and $e_i$ is incident with a vertex $u \in U$ such that there exists an edge $e_j$ incident with $u$ with $j < i$ and $x_j = 2$, then let $y_i = 1$; otherwise let $y_i = x_i$. It is clear that $r$ fixes every vertex in $\mathbb{K}$. Next we show that $(y_1, \ldots, y_m)$ is a vertex of $\mathbb{K}$: if $y_i = 0$, then $x_i = 0$; suppose $e_i$ is incident with $d \in D$ of degree larger than 1, and let $e_j$ be incident with $d$. Suppose first that $j < i$: since $y_i \neq 1$, $x_j \neq 0$, hence $y_j \neq 0$. Now suppose that $j > i$: Since $x_i = 0$, if $x_j = 0$ then $y_j = 1$, otherwise $y_j \neq 0$. The argument if $y_i = 2$ is similar. Finally we show that $r$ is edge-preserving. Suppose that $r(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$ and $r(x'_1, \ldots, x'_m) = (y'_1, \ldots, y'_m)$ are not adjacent, and hence there exists some $1 \leq i \leq m$ such that $|y_i - y'_i| \geq 2$; then without loss of generality $y_i = 0$ and $y'_i = 2$, which implies $x_i = 0$ and $x'_i = 2$ and we're done.

(2) Let $e_1, \dots, e_m$ denote the edges of $\mathbb{T}$. For each $1 \leq i \leq m$ define an integer $c_i$ as follows: remove the edge $e_i$ from $\mathbb{T}$ to obtain two connected components, exactly one of which contains a vertex $u \in U$ incident with $e_i$. Let $c_i$ denote the number of leaves of $\mathbb{T}$ that are in this component. Clearly $1 \leq c_i \leq k-2$. Viewing elements of $\mathbb{K}^0$ as columns for convenience of notation, define $f : (\mathbb{K}^0)^k \to \mathbb{K}^0$ by

$$f\left( \begin{bmatrix} x_{1,1} \\ \vdots \\ x_{m,1} \end{bmatrix} \cdots \begin{bmatrix} x_{1,k} \\ \vdots \\ x_{m,k} \end{bmatrix} \right) = \begin{bmatrix} f_1(x_{1,1}, \dots, x_{1,k}) \\ \vdots \\ f_m(x_{m,1}, \dots, x_{m,k}) \end{bmatrix}$$

where $f_i$ returns the $(c_i+1)$-th smallest entry in row $i$, i.e. if $\{x_{i,1}, \dots, x_{i,k}\} = \{u_1, \dots, u_k\}$ where $u_i \leq u_j$ when $i \leq j$ then $f_i(x_{i,1}, \dots, x_{i,k}) = u_{c_i+1}$. We must show that $f$ is a well-defined, edge-preserving NU operation.

(a) $f$ is well-defined, i.e. maps to $\mathbb{K}^0(\mathbb{T})$: indeed, let $d \in D$ of degree greater than 1 and let $e_{i_1}, \dots, e_{i_s}$ be the edges incident with $d$. Suppose for a contradiction that $f_j(x_{j,1}, \dots, x_{j,k}) \neq 0$ for all $j \in \{i_1, \dots, i_s\}$; in particular each row $x_{j,1}, \dots, x_{j,k}$ with $j \in \{i_1, \dots, i_s\}$ contains at most $c_j$ 0's. But $c_{i_1} + \cdots + c_{i_s} = k-1$, contradicting the fact that each of the $k$ columns of the matrix contains at least one entry of 0 in the relevant rows. The argument for $u \in U$ is identical, if we notice that (i) the $(c_i+1)$-th smallest entry in row $i$ equals the $(c_i'+1)$-th largest entry in row $i$ where $c_i' = k-1-c_i$, and (ii) $c_i'$ is the number of leaves of $\mathbb{T}$ that lie in the component obtained by removing $e_i$ and which contains the vertex $d \in D$ incident with $e_i$. (b) $f$ is an NU operation: indeed if $k-1$ of the columns are identical, then $2 \leq c_i + 1 \leq k-1$ implies that the $(c_i+1)$-th smallest entry in row $i$ is the repeated coordinate. (c) $f$ is edge-preserving: it is easy to see that this is equivalent to the following statement:

**Claim.** Let $2 \leq c \leq k-1$. If $|x_i - y_i| \leq 1$ for all $1 \leq i \leq k$ and $x_i \leq x_j$ when $i \leq j$, then the $c$-th smallest entry $s$ of $\{y_1, \dots, y_k\}$ satisfies $|s - x_c| \leq 1$.

To prove the claim, notice first that $y_i \leq x_i + 1 \leq x_c + 1$ for all $i \leq c$, hence at least $c$ of the $y_i$'s are at most $x_c + 1$ so we conclude that $s \leq x_c + 1$. Secondly, if $i \geq c$ we have that $y_i \geq x_i - 1 \geq x_c - 1$. Hence at least $k - c + 1$ of the $y_i$'s are at least $x_c - 1$ so $x_c - 1 \leq s$. This concludes the proof of the claim.

□

**Lemma 4.3.** *Let $\mathbb{T}$ be a tree. Then $D_r(\mathbb{T})$ is isomorphic to $\mathbb{K}(\mathbb{T})$.*

*Proof.* Let $f$ be a vertex of $D_r(\mathbb{T})$, i.e. a loop in $D(\mathbb{T})$. Recall that $f$ may be viewed as a map from $T$ to $E(\mathbb{T}) \cup T$ such that

(1) for every $t \in T$, $f(t)$ is either an edge of $\mathbb{T}$ incident with $t$ or $f(t) = t$ (if $t$ is a leaf),
(2) there is no edge $e = (s,t)$ such that $f(s) = e = f(t)$.

We assign to $f$ a tuple $\alpha(f) = (x_1, \dots, x_m)$ as follows. Let $e_i = [d,u]$ be an edge of $\mathbb{T}$ with $d \in D$ and $u \in U$. Let

$$x_i = \begin{cases} 0, & \text{if } f(d) = e_i, \\ 2, & \text{if } f(u) = e_i, \\ 1, & \text{otherwise.} \end{cases}$$

It is immediate that $\alpha(f)$ is a vertex of $\mathbb{K}(\mathbb{T})$. Let $f, g$ be vertices of $D_r(\mathbb{T})$ where $\alpha(f) = (x_1, \dots, x_m)$ and $\alpha(g) = (y_1, \dots, y_m)$: we prove that $f$ is adjacent to $g$ if and only if $\alpha(f)$ is adjacent to $\alpha(g)$. Indeed if $|x_i - y_i| \geq 2$ for some $i$ then without loss of generality $y_i = 2$ and $x_i = 0$; if $e_i = [d, u]$ this means that $f(d) = e_i = g(u)$. Conversely if $f(a) = e = g(b)$ for some edge $e_i = [a, b]$ then $|x_i - y_i| = 2$. Now let $(x_1, \dots, x_m)$ be a vertex of $\mathbb{K}(\mathbb{T})$, and define a map $f$ by the following rules: if $d \in D$ then $f(d) = e_i$ if $x_i = 0$ and $e_i$ is incident with $d$, otherwise $f(d) = (d)$; and if $u \in U$ then $f(u) = e_i$ if $x_i = 2$ and and $e_i$ is incident with $u$ otherwise $f(u) = (u)$. It is clear that $f$ is well-defined, is a loop of $D(\mathbb{T})$ and that $\alpha(f) = (x_1, \dots, x_m)$. Thus the map $\alpha$ is a graph isomorphism and we are done. □

**Corollary 4.4.** *Let $k \geq 3$, and let $\mathbb{H}$ be a connected reflexive graph. Then the following are equivalent:*

1. *$\mathbb{H}$ is a $k$-NU graph;*
2. *$\mathbb{H}$ is a retract of a product of finitely many graphs $\mathbb{K}(\mathbb{T})$ where $\mathbb{T}$ is a tree with at most $k - 1$ leaves;*
3. *$\mathbb{H}$ is a retract of a product of finitely many graphs $\mathbb{K}^0(\mathbb{T})$ where $\mathbb{T}$ is a tree with at most $k - 1$ leaves.*

*Proof.* Immediate by Lemmas 4.2, 4.3 and Theorem 3.9. □

### 4.2. **Coordinatisation by minors.**

Call a tree *reduced* if it has no vertex of degree 2.

**Definition 4.5.** *Let $\mathbb{S}$ be a reduced tree with colour classes $U$ and $D$ and edges $e_1, \dots, e_m$. Let $l_1, \dots, l_m$ be positive integers. Define a graph $\mathbb{G}(\mathbb{S}; l_1, \dots, l_m)$ as follows: its vertices are the tuples $(x_1, \dots, x_m)$ such that*

1. *$0 \leq x_i \leq l_i + 1$ for every $1 \leq i \leq m$;*
2. *for each $u \in U$ of degree greater than 1, $x_i = l_i + 1$ for exactly one edge $i$ incident with $u$;*
3. *for each $d \in D$ of degree greater than 1, $x_i = 0$ for exactly one edge $i$ incident with $d$.*

*Tuples $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ are adjacent if $|x_i - y_i| \leq 1$ for all $i$.*

*The graph $\mathbb{G}^0(\mathbb{S}; l_1, \dots, l_m)$ in defined in the same manner, except we replace "exactly one" by "at least one" in conditions (2) and (3) above.*

Notice that if $\mathbb{S}$ is an edge, then $\mathbb{G}(\mathbb{S}; l) = \mathbb{G}^0(\mathbb{S}; l)$ is a path of length $l + 1$.

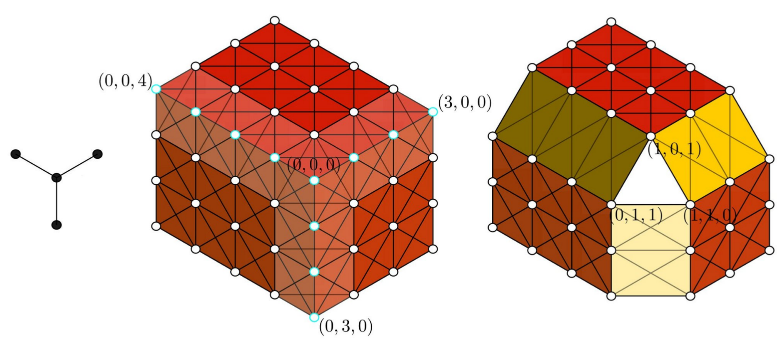


FIGURE 5. The graphs $\mathbb{G}^0(\mathbb{S}; 4, 3, 3)$ and $\mathbb{G}(\mathbb{S}; 4, 3, 3)$ for polyad $\mathbb{S}$, the star with three edges.

**Lemma 4.6.** *Let $\mathbb{S}$ be a reduced tree with $k-1$ leaves, $m$ edges and let $l_1, \ldots, l_m$ be positive integers.*

(1) $\mathbb{G}(\mathbb{S}; l_1, \ldots, l_m) \trianglelefteq \mathbb{G}^0(\mathbb{S}; l_1, \ldots, l_m)$;
(2) *the graph* $\mathbb{G}^0(\mathbb{S}; l_1, \ldots, l_m)$ *admits a $k$-ary NU polymorphism.*

*Proof.* Let $\mathbb{G} = \mathbb{G}(\mathbb{S}; l_1, \ldots, l_m)$ and $\mathbb{G}^0 = \mathbb{G}^0(\mathbb{S}; l_1, \ldots, l_m)$; let $U$ and $D$ denote the colour classes of $\mathbb{S}$.

(1) This is a very slight variation on the proof of Lemma 4.2 (1). If $(x_1, \ldots, x_m)$ is a vertex of $\mathbb{G}^0$ let $r(x_1, \ldots, x_m) = (y_1, \ldots, y_m)$ where $y_i$ is defined as follows: (a) if $x_i = 0$, and $e_i$ is incident with a vertex $d \in D$ such that there exists an edge $e_j$ incident with $d$ with $j < i$ and $x_j = 0$, let $y_i = 1$; (b) if $x_i = l_i + 1$, and $e_i$ is incident with a vertex $u \in U$ such that there exists an edge $e_j$ incident with $u$ with $j < i$ and $x_j = l_j + 1$, let $y_i = l_i$; (c) otherwise let $y_i = x_i$. The rest of the proof is identical to Lemma 4.2 (1).

(2) The proof is identical to the proof of Lemma 4.2 (2).

□

**Lemma 4.7.** *Let $\mathbb{T}$ be a tree. There exists a reduced tree $\mathbb{S}$ and positive integers $l_1, \ldots, l_m$ such that $D_r(\mathbb{T})$ is isomorphic to $\mathbb{G}(\mathbb{S}; l_1, \ldots, l_m)$.*

*Proof.* Let $\mathbb{S}$ denote the minor obtained from $\mathbb{T}$ by contracting subdivisions. Let $D$ and $U$ denote the colour classes of $\mathbb{S}$, and let $e_1, \ldots, e_m$ denote the edges of $\mathbb{S}$. Let $l_i$ be the number of edges needed to subdivide edge $e_i$ of $\mathbb{S}$ to obtain $\mathbb{T}$.

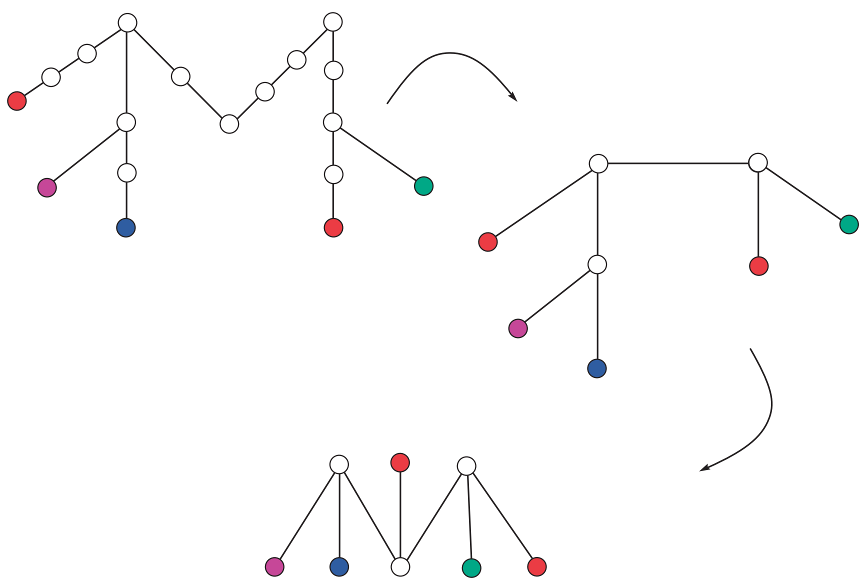

FIGURE 6. A tree $\mathbb{T}$, its minor $\mathbb{S}$, and $\mathbb{S}$ viewed as a bipartite graph.

Let $f$ be a vertex of $D_r(\mathbb{T})$, i.e. a loop in $D(\mathbb{T})$. Recall that $f$ may be viewed as a map from $T$ to $E(\mathbb{T}) \cup T$ such that

(1) for every $t \in T$, $f(t)$ is either an edge of $\mathbb{T}$ incident with $t$ or $f(t) = t$ (if $t$ is a leaf),
(2) there is no edge $e = (s, t)$ such that $f(s) = e = f(t)$.

We assign to $f$ a tuple $\alpha(f) = (x_1, \ldots, x_m)$, where $x_i$ is defined as follows. Let $e_i = [d, u]$ be an edge of $\mathbb{S}$ with $d \in D$ and $u \in U$, and denote the corresponding path in $\mathbb{T}$ by $d = v_0, v_1, \ldots, v_{l_i} = u$. For every $0 \leq j \leq l_i$ define $\beta(f, i, j) = 1$ if (a) $j = 0$ and $f(v_0) \neq [v_0, v_1]$ or (b) $j > 0$ and $f(v_j) = [v_{j-1}, v_j]$; otherwise let $\beta(f, i, j) = 0$. Notice that property (2) of $f$ implies that

$$(*) \qquad \beta(f, i, j) \geq \beta(f, i, j') \text{ whenever } j \leq j'.$$

Define $x_i = \sum_{j=0}^{l_i} \beta(f, i, j)$. Roughly speaking, $x_i$ counts the number of vertices $v_i$ that $f$ "maps to the left" along the path corresponding to edge $e_i$. It is clear that $0 \leq x_i \leq l_i + 1$. If $d$ has degree greater than 1, then $f(d)$ is some edge incident to $d$ and hence $\beta(f, n, 0) = 1$ for all but one value of $n$; in particular there is at most one index $n$ such that $x_n = 0$ with $e_n$ incident to $d$. If $\beta(f, i, 0) = 0$ then by (*) $\beta(f, i, j) = 0$ for all $j$ so $x_i = 0$. If $u$ has degree greater than 1, then $f(u)$ is some edge incident to $u$ and hence $\beta(f, n, l_n) = 1$ for at most one value of $n$; in particular there is at most one index $n$ such that $x_n = l_n + 1$ with $e_n$ incident to $u$. If $\beta(f, i, l_i) = l_i + 1$ then by (*) $\beta(f, i, j) = 1$ for all $j$ so $x_i = l_i + 1$.

Let $f, g$ be vertices of $D_r(\mathbb{T})$ and let $\alpha(f) = (x_1, \ldots, x_m)$ and let $\alpha(g) = (y_1, \ldots, y_m)$: we prove that $f$ is adjacent to $g$ if and only if $\alpha(f)$ is adjacent to $\alpha(g)$. Suppose first that $\alpha(f)$ and $\alpha(g)$ are not adjacent. Then $|y_i - x_i| \geq 2$ for some $i$, and suppose without loss of generality that $y_i > x_i$. Let $j_0$ be the largest index such that $\beta(g, i, j_0) = 1$. Since $y_i - x_i > 1$ there exists some $j < j_0$ such that $\beta(f, i, j) = 0$, which means we can find an edge $e = (a, b)$ on the path corresponding to edge $e_i$ such that $f(a) = e = g(b)$ and we're done. Conversely, suppose that $f$ and $g$ are not adjacent in $D_r(\mathbb{T})$; then there exists an edge $e = (a, b)$ such that $f(a) = e = f(b)$. Edge $e$ lies on the path corresponding to some edge $e_i$ of $\mathbb{S}$, so that $a = v_j$ and $b = v_{j+1}$ with the notation as above. Then $\beta(f, i, j) = 0$ and $\beta(g, i, j+1) = 1$, which by (*) implies that $x_i \leq j-1$ and $y_i \geq j+1$ so $|y_i - x_i| \geq 2$.

It remains to show that $\alpha$ is a bijection. Let $(x_1, \ldots, x_m)$ be a vertex of $\mathbb{G}(\mathbb{S}; l_1, \ldots, l_m)$. We construct a map $f$ such that $\alpha(f) = (x_1, \ldots, x_m)$. Let $v \in T$. Suppose first that $v \in S$ is of degree 1 in $\mathbb{S}$. Then there exists a unique edge $e_i$ incident to $v$; let $v_0, \ldots, v_{l_i}$ be the path in $\mathbb{T}$ that corresponds to $e_i$. If $v \in D$ and $x_i = 0$ let $f(v) = [v_0, v_1]$, and if $x_i > 0$ let $f(v) = (v)$. If $v \in U$ and $x_i = l_i + 1$ then let $f(v) = [v_{l_i - 1}, v_{l_i}]$ and if $x_i < l_i$ let $f(v) = (v)$. Now suppose that $v \in S$ has degree greater than 1. Then there exists a unique edge $e_i$ incident to $v$ such that $x_i = 0$ if $v \in D$ and $x_i = l_i + 1$ if $v \in U$. Let $v_0, \ldots, v_{l_i}$ be the path in $\mathbb{T}$ that corresponds to $e_i$. If $v \in D$ let $f(v) = [v_0, v_1]$, and if $v \in U$ let $f(v) = [v_{l_i - 1}, v_{l_i}]$. Finally suppose that $v \notin S$, so there exists an index $0 < j < l_i$ such that $v = v_j$ lies on the path $v_0, \ldots, v_{l_i}$ that corresponds to some edge $e_i$ of $\mathbb{S}$. If $x_i \geq j + 1$ then let $f(v) = [v_{j-1}, v_j]$, otherwise let $f(v) = [v_j, v_{j+1}]$. It is easy to verify that the resulting map $f$ satisfies conditions (1) and (2) above, and that $\alpha(f) = (x_1, \ldots, x_m)$. □

**Corollary 4.8.** *Let $k \geq 3$, and let $\mathbb{H}$ be a connected reflexive graph. Then the following are equivalent:*

(1) *$\mathbb{H}$ is a $k$-NU graph;*

(2) $\mathbb{H}$ *is a retract of a product of finitely many graphs of the form* $\mathbb{G}(\mathbb{S}; l_1, \ldots, l_m)$ *where* $\mathbb{S}$ *has at most* $k-1$ *leaves;*
(3) $\mathbb{H}$ *is a retract of a product of finitely many graphs of the form* $\mathbb{G}^0(\mathbb{S}; l_1, \ldots, l_m)$ *where* $\mathbb{S}$ *has at most* $k-1$ *leaves.*

*Proof.* Immediate by Lemmas 4.6, 4.7 and Theorem 3.9. □

**Remark.** Notice that if the tree $\mathbb{T}$ is a path then the minor $\mathbb{S}$ obtained by contracting subdivisions is an edge, and hence the 3-ary NU operation we defined in Lemma 4.6 (2) is the standard majority operation that returns the middle element. More generally, if $\mathbb{T}$ is a polyad, then $\mathbb{S}$ is a star and the NU operation we defined returns the second smallest element in each row, a natural generalisation of the path case.

## 5. Absolute Retracts

Let $\mathbb{H}$ be a digraph and let $u, v \in H$. We let $d_{\mathbb{H}}(u,v)$ denote the usual graph distance between $u$ and $v$ in $\mathbb{H}^u$, i.e. the length of a shortest path between $u$ and $v$. If $f : \mathbb{G} \to \mathbb{H}$ is a homomorphism then $d_{\mathbb{H}}(f(u), f(v)) \le d_{\mathbb{G}}(u,v)$ for all $u, v \in G$, i.e. $f$ is a *non-expansive* map. A distance-preserving map is called an *isometry.* If $e : \mathbb{G} \hookrightarrow \mathbb{H}$ is a one-to-one isometry from the graph $\mathbb{G}$ to the graph $\mathbb{H}$, we say that $\mathbb{G}$ is *isometrically embedded* in $\mathbb{H}$. A graph $\mathbb{H}$ is an *absolute retract (for isometry)* if it is a retract of every graph into which it isometrically embeds (see [14],[18],[15],[16]).

We begin by stating the basic definitions we need.

### 5.1. Absolute Retracts and Polyad Duality.

**Definition 5.1.** [14, 18] *Let* $\mathbb{H}$ *be a graph and let* $k \ge 2$*. A* $k$-hole *in* $\mathbb{H}$ *is a pair* $(L, f)$ *where* $L \subseteq H$ *is a* $k$*-element set of vertices of* $\mathbb{H}$ *and* $f : L \to \mathbb{Z}^+$ *such that*

(1) *no vertex* $x \in H$ *satisfies* $d(x,l) \le f(l)$ *for all* $l \in L$*,*
(2) *for every proper subset* $L' \subset L$*, there exists* $x \in H$ *that satisfies* $d(x,l) \le f(l)$ *for all* $l \in L'$*.*

**Definition 5.2.** [14, 18] *Suppose* $\mathbb{G}$ *is an induced subgraph of* $\mathbb{H}$ *and let* $(L, f)$ *be a* $k$*-hole of* $\mathbb{G}$*. We say that* $(L, f)$ *is* filled *in* $\mathbb{H}$ *if there exists some* $v \in H$ *such that* $d(v,l) \le f(l)$ *for all* $l \in L$*. We say that* $(L, f)$ *is* separated *in* $\mathbb{H}$ *if it is not filled in* $\mathbb{H}$*. Let* $j \ge 2$*. The graph* $\mathbb{G}$ *is a* $j$-absolute retract *if it is a retract of any graph* $\mathbb{H}$ *into which it embeds such that every* $j'$*-hole is separated, for all* $j' \le j$*. For each* $j \ge 2$ *let* $\mathscr{AR}_j$ *denote the class of* $j$-absolute retracts*, and let*

$$\mathscr{AR} = \bigcup_{k \ge 2} \mathscr{AR}_k$$

*denote the class of* absolute retracts *(see Definition 5.11 and Lemma 5.13 below for concrete examples).*

In this section we use our results to describe very simple generating sets for the varieties $\mathscr{AR}_k$ ($k \ge 2$). The overall strategy is to interpret notions such as holes and absolute retracts in terms of very special trees called *polyads*, and their duals. We will deduce immediately that the variety $\mathscr{AR}_3$ coincides with the class of reflexive graphs admitting a 4-ary NU, answering a long-standing open question (it is stated in [6] but certainly it is much older).

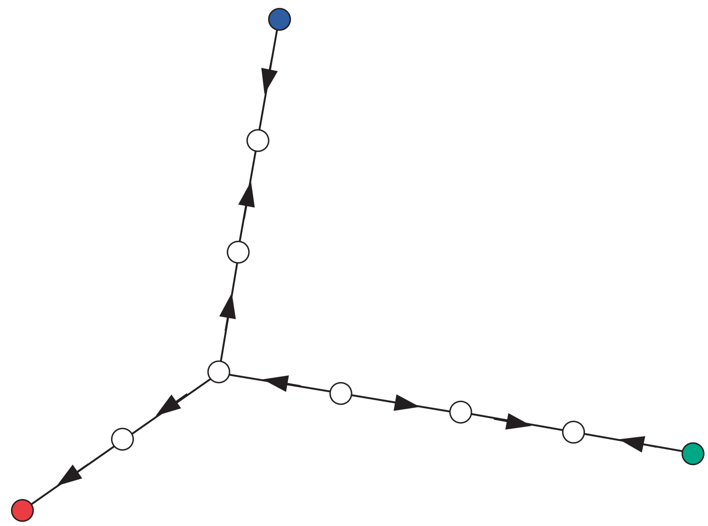

FIGURE 7. A non-degenerate 3-coloured polyad.

**Definition 5.3.** *Let $\mathbb{H}$ be a graph, let $\mathbb{T}$ be an elementary $\tau_{\mathbb{H}}$-tree and let $k \geq 1$. We say that $\mathbb{T}$ is a $k$-*coloured $\tau_{\mathbb{H}}$-polyad *if it is of one of the following forms:*

(1) *($k = 1$) a single vertex with two distinct labels;*
(2) *($k = 2$) an edge with both ends coloured; or*
(3) *($k \geq 2$) $\mathbb{T}$ has one non-coloured vertex of degree $k$ called its* central vertex*, all other non-coloured vertices are of degree 2, and its $k$ coloured vertices are of degree 1. Each path connecting the central vertex to a coloured vertex is called a* branch*.*

The polyads of form (1) or (2) we shall call *degenerate* while those of form (3) will be called *non-degenerate*. Notice that the 2-coloured polyads are exactly the paths with coloured extremities, and that the non-degenerate ones are those of length at least 2.

Our first goal is to prove that the members of $\mathscr{A}\mathscr{R}_k$ are precisely those graphs $\mathbb{H}$ such that the structure $\mathbb{H}^c$ admits a (finite) duality consisting only of $m$-coloured polyads with $m \leq k$.

Let $\mathbb{H}$ be a graph, let $k \geq 2$ and let $\mathcal{L} = (L, f)$ be such that $L = \{c_1, \dots, c_k\} \subseteq H$ is a $k$-element set of vertices of $\mathbb{H}$ and $f : L \to \mathbb{Z}^+$. Construct the $k$-coloured $\tau_{\mathbb{H}}$-polyad $T_{\mathcal{L}}$ as follows: for each $1 \leq i \leq k$ let $T_{\mathcal{L}}$ have a branch of length $f(c_i)$ with endpoint coloured by $c_i$.

Let $\mathbb{T}$ be a non-degenerate $k$-coloured $\tau_{\mathbb{H}}$-polyad, and assume that its coloured elements carry labels $c_1, \dots, c_k$ that are all distinct. Define $\mathcal{L}_{\mathbb{T}} = (L, f)$ where $L = \{c_1, \dots, c_k\}$ and $f(c_i)$ is equal to the (undirected) distance of the vertex coloured $c_i$ to the polyad's central vertex.

**Definition 5.4.** *Let $\mathbb{H}$ be an induced subgraph of $\mathbb{K}$. Let $\mathbb{K}_{\mathbb{H}}$ denote the $\tau_{\mathbb{H}}$-structure that consists of the graph $\mathbb{K}$ with every vertex $h \in H$ coloured by $\{h\}$.*

**Lemma 5.5.** *Let $\mathbb{H}$ be a graph, let $k \geq 2$. Let $\mathbb{T}$ be a non-degenerate $k$-coloured $\tau_{\mathbb{H}}$-polyad and let $\mathcal{L} = (L, f)$ where $L \subseteq H$ and $f : L \to \mathbb{Z}^+$. Then*

1. *$\mathcal{L}_{(\mathbb{T}_{\mathcal{L}})} = \mathcal{L}$; if $\mathbb{T}$ has $k$ distinct colours then $\mathbb{T}_{(\mathcal{L}_{\mathbb{T}})} = \mathbb{T}$;*
2. *if $\mathbb{H}$ is an induced subgraph of $\mathbb{K}$ and $\mathcal{L}$ is a $k$-hole in $\mathbb{H}$ then $\mathbb{K}$ fills the hole $\mathcal{L}$ if and only if $\mathbb{T}_{\mathcal{L}} \to \mathbb{K}_{\mathbb{H}}$;*
3. *$\mathcal{L}$ is a $k$-hole of $\mathbb{H}$ if and only if $\mathbb{T}_{\mathcal{L}}$ is a critical obstruction of $\mathbb{H}^c$;*
4. *if $\mathbb{T}$ is a critical obstruction of $\mathbb{H}^c$, then $\mathcal{L}_{\mathbb{T}}$ is well-defined and is a $k$-hole of $\mathbb{H}$;*
5. *if $\mathcal{L}_{\mathbb{T}}$ is a $k$-hole of $\mathbb{H}$ then $\mathbb{T}$ is a critical obstruction of $\mathbb{H}^c$.*

*Proof.* (1): Straightforward.

(2): ($\Leftarrow$) Let $\phi : \mathbb{T}_{\mathcal{L}} \to \mathbb{K}_{\mathbb{H}}$ and let $x$ denote the central vertex of $\mathbb{T}_{\mathcal{L}}$. It is immediate that $d(\phi(x), c_i) \leq f(c_i)$ for all $i$, and hence $\mathcal{L}$ is filled in $\mathbb{K}$.

($\Rightarrow$) Suppose that $u \in \mathbb{K}$ satisfies $d(u, c_i) \leq f(c_i)$ for all $i$. Because the target graph is reflexive, each branch of $\mathbb{T}_{\mathcal{L}}$ can be mapped to $\mathbb{K}_{\mathbb{H}}$ sending the central vertex to $u$ and the other endpoint to $c_i$, hence $\mathbb{T}_{\mathcal{L}} \to \mathbb{K}_{\mathbb{H}}$.

(3): ($\Rightarrow$) If $\mathcal{L}$ is a $k$-hole of $\mathbb{H}$ then by (2) there cannot be a homomorphism from $\mathbb{T}_{\mathcal{L}}$ to $\mathbb{H}^c = \mathbb{H}_{\mathbb{H}}$. Now if we remove any edge or colour of $\mathbb{T}_{\mathcal{L}}$, say it is on the branch $P_j$ with endpoint coloured $c_j$, then we can find a homomorphism from the resulting structure $\mathbb{T}'$ to $\mathbb{H}^c$. Indeed, by minimality property of a $k$-hole we can find a vertex $y \in H$ such that $d(y, c_i) \leq f(c_i)$ for all $i \neq j$. This defines an obvious homomorphism (use the fact that the graph $\mathbb{H}$ is reflexive.)

($\Leftarrow$) Suppose that $\mathbb{T}_{\mathcal{L}}$ is a critical obstruction of $\mathbb{H}^c$. Since $\mathbb{T}_{\mathcal{L}} \not\to \mathbb{H}^c$, there can obviously not be any $y \in H$ such that $d(y, c_i) \leq f(c_i)$ for all $i$. If $L'$ is a proper subset of $L$, say without loss of generality $L' = \{c_1, \ldots, c_m\}$, then remove from $\mathbb{T}_{\mathcal{L}}$ all colours $c_j$ with $j > m$; we can map the resulting structure to $\mathbb{H}^c$ and the value $y$ on the central vertex will satisfy $d(v, c_i) \leq f(c_i)$ for all $i \leq m$.

(4): If $\mathbb{T}$ is a critical obstruction then it has $k$ distinct colours, for otherwise we could erase a repeated colour $c$ (choosing the longest path labelled with it), and map to $\mathbb{H}^c$; now simply use the values on the shorter path to a colour $c$ to redefine the map on the unlabelled path to obtain a map from $\mathbb{T}$ to $\mathbb{H}^c$, a contradiction. Thus $\mathcal{L}_{\mathbb{T}}$ is well-defined. Now by (1) we have that $\mathbb{T} = \mathbb{T}_{(\mathcal{L}_{\mathbb{T}})}$; hence by (3) $\mathcal{L}_{\mathbb{T}}$ is a $k$-hole of $\mathbb{H}$.

(5): By (1) we have that $\mathbb{T} = \mathbb{T}_{(\mathcal{L}_{\mathbb{T}})}$; so if $\mathcal{L}_{\mathbb{T}}$ is a $k$-hole of $\mathbb{H}$, then by (3) $\mathbb{T}$ is a critical obstruction of $\mathbb{H}^c$.

□

To prove the main result of this section, we also require a well-known (folklore) construction which is used to show the equivalence of the retraction problem on the structure $\mathbb{H}$ to $CSP(\mathbb{H}^c)$. Let $\mathbb{H}$ be a graph and let $\mathbb{K}$ be a $\tau_{\mathbb{H}}$-structure. We construct (in a straightforward manner) a graph $\mathbb{K}'$ that contains $\mathbb{H}$ as an induced subgraph such that $\mathbb{K}'$ retracts onto $\mathbb{H}$ if and only if $\mathbb{K} \to \mathbb{H}$. The trick is essentially to adjoin a disjoint copy of $\mathbb{H}$ to the underlying graph of $\mathbb{K}$, and then to identify each vertex $h \in H$ with the vertices of $\mathbb{K}$ coloured by $\{h\}$. Notice first that if some vertex of $\mathbb{K}$ is coloured by more than one colour then $\mathbb{K} \not\to \mathbb{H}$ so we may assume this does not occur. Similarly, we may assume that if $u$ and $v$ are adjacent coloured vertices in $\mathbb{K}$ then their colours are adjacent in $\mathbb{H}$. Hence we may assume that the colouring of $\mathbb{K}$ is a partial graph homomorphism $f$ defined on some subset $C$ of $K$ (the coloured elements of $\mathbb{K}$) to $\mathbb{H}$. Construct a new graph $\mathbb{K}'$ as follows: let $\mathbb{X}$ be the disjoint union of $\mathbb{H}$ and the underlying graph of $\mathbb{K}$. Extend the map $f$ to a self-map $F$ (it is NOT a homomorphism !) on $X$ by defining $F(x) = x$ for all $x \in X \setminus C$. Let $\theta$ denote the kernel of $F$, i.e. $(x, y) \in \theta$ if and only if $F(x) = F(y)$. Clearly $\theta$ is an equivalence relation, which decomposes the set $X$ into disjoint *blocks*. The vertices of $\mathbb{K}'$ are these blocks, and blocks $A$ and $B$ are adjacent if there exist $a \in A$ and $b \in B$ that are adjacent in $\mathbb{X}$. If $x \in X$ let $[x]$ denote its block. It is clear that $\mathbb{K}'$ contains a copy $\mathbb{H}'$ of the graph $\mathbb{H}$ (since $f$ is a graph homomorphism no new edges are created) that consists of the $[h]$ with $h \in H$.

**Lemma 5.6.** *Let $\mathbb{H}$ be a graph and let $\mathbb{K}$ be a $\tau_{\mathbb{H}}$-structure. Then $\mathbb{H}' \trianglelefteq \mathbb{K}'$ if and only if $\mathbb{K} \to \mathbb{H}^c$.*

*Proof.* ($\Leftarrow$) Let $\phi : \mathbb{K} \to \mathbb{H}^c$. Since $\phi$ preserves colours it is clear that it is constant on blocks of $\theta$ so we may define $r : \mathbb{K}' \to \mathbb{H}'$ by $r([x]) = [\phi(x)]$. It is easy to see that this is a homomorphism and a retraction.

($\Rightarrow$) Let $r : \mathbb{K}' \to \mathbb{H}'$ be a retraction. Define $\phi : \mathbb{K} \to \mathbb{H}^c$ by setting $\phi(x) = h$ where $[h] = r([x])$. If $f(x) = h$ then $r([x]) = r([h]) = [h]$ since $r$ is a retraction so $\phi$ preserves colours, and clearly $\phi$ preserves edges. □

**Theorem 5.7.** *Let $\mathbb{H}$ be a connected graph and let $k \geq 2$. Then $\mathbb{H} \in \mathscr{AR}_k$ if and only if $\mathbb{H}^c$ has a (finite) duality consisting of $m$-coloured $\tau_{\mathbb{H}}$-polyads, $m \leq k$.*

*Proof.* ($\Leftarrow$) Suppose that $\mathbb{H}^c$ has a (finite) duality $\mathcal{D}$ consisting of $m$-coloured $\tau_{\mathbb{H}}$-polyads, $m \leq k$. Suppose that one of these polyads is not a critical obstruction: it is easy to see that it then must contain a critical obstruction which is itself a polyad. Indeed, notice first that a non-critical polyad must be non-degenerate. If removing a colour or an edge along some branch leaves an obstruction, then deletion of the whole branch leaves a smaller polyad obstruction. Hence we may suppose without loss of generality that every polyad in the duality is in fact critical. Suppose that $\mathbb{H}$ is an induced subgraph of $\mathbb{K}$ and that $\mathbb{H}$ is not a retract of $\mathbb{K}$. Then $\mathbb{K}_{\mathbb{H}} \not\to \mathbb{H}^c$, so there exists a $\mathbb{T} \in \mathcal{D}$ such that $\mathbb{T} \to \mathbb{K}_{\mathbb{H}}$. By definition of $\mathbb{K}_{\mathbb{H}}$ it is clear that $\mathbb{T}$ is non-degenerate. Since $\mathbb{T}$ is a critical obstruction, by Lemma 5.5 (4) $\mathcal{L}_{\mathbb{T}}$ is an $m$-hole for $\mathbb{H}$ with $m \leq k$, so by Lemma 5.5 (1) and (2) $\mathcal{L}_{\mathbb{T}}$ is filled in $\mathbb{K}$.

($\Rightarrow$) Suppose now that $\mathbb{H} \in \mathscr{AR}_k$. Let $\mathcal{D} = \{\mathbb{T}_1, \dots, \mathbb{T}_s\}$ be the set of all critical obstructions of $\mathbb{H}^c$ that are $m$-coloured polyads with $m \leq k$; notice that there are indeed finitely many of these, since $\mathbb{H}$ is connected we can find a vertex which is at finite distance from any given set of vertices and hence the number of critical polyad obstructions of $\mathbb{H}$ is finite. We want to show that $\mathcal{D}$ is a duality for $\mathbb{H}^c$. So let $\mathbb{K}$ be a $\tau_{\mathbb{H}}$-structure that does not admit a homomorphism to $\mathbb{H}^c$; we must show there exists some $\mathbb{T} \in \mathcal{D}$ such that $\mathbb{T} \to \mathbb{K}$.

If a degenerate polyad in $\mathcal{D}$ maps to $\mathbb{K}$ we are done. Otherwise, we have that that (i) no coloured vertex of $\mathbb{K}$ receives more than one colour, and (ii) the partial map sending a coloured vertex to its colour is a partial homomorphism. Thus we may construct the graph $\mathbb{K}'$ as described above; by Lemma 5.6 $\mathbb{K}'$ does not admit a retraction to its subgraph $\mathbb{H}'$ isomorphic to $\mathbb{H}$. Since $\mathbb{H} \in \mathscr{AR}_k$ there exists some $m$-hole $\mathcal{L}$ ($m \leq k$) of $\mathbb{H}$ that is filled in $\mathbb{K}'$; By Lemma 5.5 (2) the associated $m$-coloured polyad $\mathbb{T} = \mathbb{T}_{\mathcal{L}}$ admits a homomorphism $\phi$ to $\mathbb{K}'_{\mathbb{H}}$. We may suppose that, of all critical $m$-coloured ($m \leq k$) polyad obstructions of $\mathbb{H}$ that admit a homomorphism to $\mathbb{K}'_{\mathbb{H}}$, $\mathbb{T}$ has the smallest set of vertices. We show that $\mathbb{T} \to \mathbb{K}$.

**Claim.** $\phi(u)$ is non-coloured for every non-coloured vertex $u$ of $\mathbb{T}$.

*Proof of Claim.* Indeed, suppose that some non-coloured vertex $u$ of $\mathbb{T}$ is mapped via $\phi$ to some coloured vertex of $\mathbb{K}'_{\mathbb{H}}$, say of colour $c$. If $u$ is the central vertex of the polyad, then certainly one branch cannot be mapped to $\mathbb{H}$ sending $u$ to $c$ (otherwise we could map $\mathbb{T}$ to $\mathbb{H}$). Hence this defines a smaller critical polyad obstruction which maps to $\mathbb{K}'_{\mathbb{H}}$ via $\phi$, contradicting the minimality of $\mathbb{T}$. Otherwise $u$ is of degree 2 and its removal creates two components. Define two new polyads $\mathbb{T}_1$ and $\mathbb{T}_2$ in the obvious way from these components (with $u$ added) by colouring vertex $u$ by colour $c$.

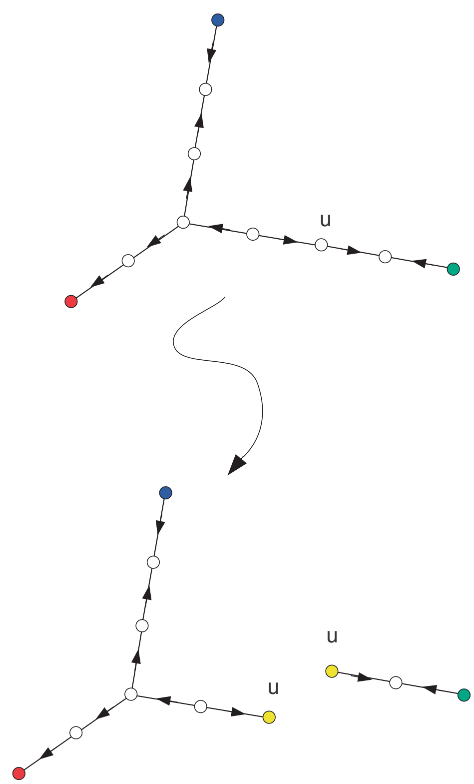


FIGURE 8. The polyads $\mathbb{T}$, $\mathbb{T}_1$ and $\mathbb{T}_2$.

Obviously one of these cannot map to $\mathbb{H}$, say $\mathbb{T}_1$, hence it contains a critical obstruction of $\mathbb{H}$ which is an $m$-coloured polyad, $m \leq k$ which maps to $\mathbb{K}'_{\mathbb{H}}$ via $\phi$, another contradiction, which concludes the proof of the claim.

Recall that the non-coloured elements of $\mathbb{K}'_{\mathbb{H}}$ are one-element blocks containing a non-coloured vertex of $\mathbb{K}$, so by the claim $\phi$ defines naturally a homomorphism $\phi'$ from the set of non-coloured vertices of $\mathbb{T}$ to $\mathbb{K}$. Now let $v$ be a coloured vertex of $\mathbb{T}$, with colour $c$: it is adjacent to a unique non-coloured vertex $u$ of $\mathbb{T}$. Since $\phi(u) = \{z\}$ being adjacent to $c$ in $\mathbb{K}'_{\mathbb{H}}$ means that $z$ is adjacent in $\mathbb{K}$ to some vertex

$w$ which is coloured by $c$, let $\phi'(v) = w$. Then $\phi'$ is the desired homomorphism from $\mathbb{T}$ to $\mathbb{K}$. □

5.2. **A generating set for $\mathscr{AR}_k$.** Following the remark we made before Theorem 3.6, we shall call a tree $\mathbb{T}$ a *(non-degenerate) polyad* if there exists a reflexive graph $\mathbb{H}$ and a (non-degenerate) $\tau_{\mathbb{H}}$-polyad $\mathbb{S}$ such that $(\mathbb{S}^\tau)^u$ is isomorphic to $\mathbb{T}$, and use $D_r(\mathbb{T})$ to denote $D_r(\mathbb{S})$.

**Definition 5.8.** *Let $k \geq 2$ and $\lambda \geq 1$ be integers. Let $\mathbb{D}^k(\lambda)$ denote the graph whose vertices are all $k$-tuples $(x_1, \ldots, x_k)$ with $0 \leq x_i \leq \lambda+1$ for all $i$ that contain exactly one entry equal to 0; two tuples $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are adjacent precisely if $|x_i - y_i| \leq 1$ for all $1 \leq i \leq k$.*

We shall show that the graphs $\mathbb{D}^k(\lambda)$ with $\lambda \geq 1$ generate the variety $\mathscr{AR}_k$.

**Lemma 5.9.** *Let $\mathbb{T}$ be a path of length $l$, let $\lambda$ be a positive integer, and let $2 \leq m \leq k$.*

1. *$D_r(\mathbb{T})$ is a path of length $l+1$;*
2. *$\mathbb{D}^2(\lambda)$ is a path of length $2\lambda+1$;*
3. *$\mathbb{D}^m(\lambda) \trianglelefteq \mathbb{D}^k(\lambda)$.*

*Proof.* (1) If $\mathbb{T}$ is a path, it follows from the proof of Lemma 4.7 that $D_r(\mathbb{T})$ is isomorphic to $\mathbb{G}(\mathbb{S}; l)$ where $\mathbb{S}$ is an edge, and this graph is obviously a path of length $l+1$. (2) This is a straightforward verification. (3) It clearly suffices to prove the result for $k = m+1$. Define a map $e : \mathbb{D}^m(\lambda) \to \mathbb{D}^{m+1}(\lambda)$ as follows: $e(x_1, \ldots, x_m) = (x_1, \ldots, x_m, y)$ where

$$y = \begin{cases} x_m, & \text{if } x_m \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to verify that $e$ is an edge-preserving embedding of $\mathbb{D}^m(\lambda)$ into $\mathbb{D}^{m+1}(\lambda)$.

Next define $r : \mathbb{D}^{m+1}(\lambda) \to \mathbb{D}^m(\lambda)$ as follows: $r(y_1, \ldots, y_{m+1}) = (x_1, \ldots, x_m)$ where

$$x_i = \begin{cases} \min(y_m, y_{m+1}), & \text{if } i = m, \\ y_i, & \text{otherwise.} \end{cases}$$

It is not difficult to verify that $r$ is a well-defined edge-preserving map from $\mathbb{D}^{m+1}(\lambda)$ to $\mathbb{D}^m(\lambda)$ and that $r \circ e$ is the identity map on $\mathbb{D}^m(\lambda)$. □

**Lemma 5.10.** *Let $k \geq 2$ and let $\mathbb{H} \in \mathscr{AR}_k$. Then there exists $\lambda, s \geq 1$ such that $\mathbb{H} \trianglelefteq (\mathbb{D}^k(\lambda))^s$.*

*Proof.* **Claim 1.** There exist non-degenerate polyads $\mathbb{T}_1, \ldots, \mathbb{T}_s$ with at most $k$ leaves such that

$$\mathbb{H} \trianglelefteq \prod_{i=1}^{s} D_r(\mathbb{T}_i).$$

*Proof of Claim 1.* By Theorem 5.7, if $\mathbb{H} \in \mathscr{AR}_k$ then it has a duality $\{\mathbb{T}_1, \ldots, \mathbb{T}_s\}$ where each $\mathbb{T}_i$ is an $m$-coloured $\tau_{\mathbb{H}}$-polyad with $m \leq k$. Since subtrees of polyads are polyads, by the proof of Theorem 3.9 we can express $\mathbb{H}$ as a retract of a product of graphs $D_r(\mathbb{T}_i)$ where the $\mathbb{T}_i$ are $m$-coloured $\tau_{\mathbb{H}}$-polyads, $m \leq k$. It is easy to see that if $\mathbb{T}$ is a degenerate polyad then its reflexive dual is a path of length 2 or 3; by Lemma 5.9 (1) the reflexive dual of a non-degenerate polyad with 2 leaves is a path of length at least 3, and since shorter paths are retracts of it we may do without them in the representation.

**Claim 2.** Let $m \geq 2$, let $l_1, \dots, l_m$ be positive integers, let $\lambda = \max\{l_1, \dots, l_m\}$, and let $\mathbb{T}$ be a (non-degenerate) polyad with $m$ branches of lengths $l_1, \dots, l_m$ respectively. Then $D_r(\mathbb{T}) \trianglelefteq \mathbb{D}^m(\lambda)$.

*Proof of Claim 2.* Suppose that $m = 2$: then $\mathbb{T}$ is a path of length $l_1 + l_2$. By Lemma 5.9, $D_r(\mathbb{T})$ is a path of length $l_1 + l_2 + 1$ and $\mathbb{D}^2(\lambda)$ is a path of length $2\lambda + 1 \geq l_1 + l_2 + 1$ hence $D_r(\mathbb{T}) \trianglelefteq \mathbb{D}^2(\lambda)$ and the result follows.

Now suppose that $m \geq 3$. Let $\mathbb{S}$ be the star with $m$ branches, i.e. the polyad with $m$ branches of length 1. By the proof of Lemma 4.7 $D_r(\mathbb{T})$ is isomorphic to $\mathbb{G}(\mathbb{S}; l_1, \dots, l_m)$ which consists of all tuples $(x_1, \dots, x_m)$ with $0 \leq x_i \leq l_i + 1$ with a unique $j$ such that $x_j = 0$ (choose $D$ to be the central vertex and $U$ the leaves of $\mathbb{S}$). Obviously $\mathbb{G} = \mathbb{G}(\mathbb{S}; l_1, \dots, l_m)$ is an induced subgraph of $\mathbb{M} = \mathbb{D}^m(\lambda)$; define a map $r : \mathbb{M} \to \mathbb{G}$ by

$$r(x_1, \dots, x_m) = (y_1, \dots, y_m)$$

where $y_i = \min(x_i, l_i)$. It is easy to verify that this is a homomorphism (it clearly is a retraction.) This completes the proof of the claim.

Notice that the last argument also shows that $\mathbb{D}^k(\lambda) \trianglelefteq \mathbb{D}^k(\lambda')$ if $\lambda \leq \lambda'$. The result now follows easily from this observation and Claims 1 and 2. □

The next three lemmas are devoted to the proof that the graphs $\mathbb{D}^k(\lambda)$ are themselves absolute retracts.

**Definition 5.11.** *Let $\lambda \geq 1$ and let $k \geq 2$ and consider the following graph $\mathbb{R}^k(\lambda)$: its vertices are the tuples $(x_1, \dots, x_k)$ that satisfy the following conditions:*

(1) *$0 \leq x_i \leq 3\lambda + 3$ for all $i$;*
(2) *$x_i + x_j \geq 2\lambda + 2$ for all $i < j$;*
(3) *there exists some $j$ such that $x_j \geq 2\lambda + 2$.*

*Tuples $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ are adjacent precisely if $|x_i - y_i| \leq 1$ for all $1 \leq i \leq k$.*

**Lemma 5.12.** *For every $\lambda \geq 1$ and $k \geq 2$, $\mathbb{D}^k(\lambda) \trianglelefteq \mathbb{R}^k(\lambda)$.*

*Proof.* One verifies immediately that the map $e : \mathbb{D}^k(\lambda) \to \mathbb{R}^k(\lambda)$ defined by $e(x_1, \dots, x_k) = (2\lambda + 2 - x_1, \dots, 2\lambda + 2 - x_k)$ is a graph embedding and that its image $S$ consists of all tuples $(y_1, \dots, y_k)$ that satisfy (i) $\lambda + 1 \leq y_i \leq 2\lambda + 2$ for all $i$ and (ii) there exists a unique $j$ such that $y_j = 2\lambda + 2$. Thus $\mathbb{D}^k(\lambda)$ is isomorphic to the subgraph of $\mathbb{R}^k(\lambda)$ induced by $S$.

We define our retraction $r$ onto $S$ as follows: if $(x_1, \dots, x_k)$ is a vertex of $\mathbb{R}^k(\lambda)$, let $x_{i_1}, \dots, x_{i_p}$ (where $i_1 \leq \cdots \leq i_p$) denote its coordinates that are at least $2\lambda + 2$ (so that $p \geq 1$), and let $x_{j_1}, \dots, x_{j_q}$ (where $j_1 \leq \cdots \leq j_q$) denote its coordinates that are at most $\lambda$ (so that $q \leq 1$). Define $r(x_1, \dots, x_k)$ as the tuple obtained from $(x_1, \dots, x_k)$ by replacing

- each of $x_{i_1}, \dots, x_{i_{p-1}}$ by $2\lambda + 1$ (if $p \geq 2$),
- $x_{i_p}$ by $2\lambda + 2$, and
- each of $x_{j_1}, \dots, x_{j_q}$ by $\lambda + 1$ (if $q \geq 1$).

Suppose that $(x_1, \dots, x_k) \in S$: then it is clear that $r$ fixes it. It is also clear that $r$ maps any tuple of $\mathbb{R}^k(\lambda)$ into $S$. It remains to show that $r$ is a graph homomorphism. Let $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ be adjacent in $\mathbb{R}^k(\lambda)$ and let $(x'_1, \dots, x'_k)$

and $(y'_1, \dots, y'_k)$ be their respective images under $r$. Fix some coordinate $i$ such that, without loss of generality, $x_i \neq x'_i$. There are several cases:

(1) Case 1: $x_i \geq 2\lambda{+}2$ and $x'_i = 2\lambda{+}1$. Then $y_i \geq 2\lambda{+}1$ so $y'_i \in \{2\lambda{+}1, 2\lambda{+}2\}$.
(2) Case 2: $x_i > 2\lambda{+}2$ and $x'_i = 2\lambda{+}2$. Then $y_i \geq 2\lambda{+}2$ so $y'_i \in \{2\lambda{+}1, 2\lambda{+}2\}$.
(3) Case 3: $x_i \leq \lambda$, $x'_i = \lambda + 1$. Then $y_i \leq \lambda + 1$ so $y'_i = \lambda + 1$.

□

**Lemma 5.13.** *For every $\lambda \geq 1$ and $k \geq 2$, $\mathbb{R}^k(\lambda) \in \mathscr{A}\mathscr{R}_k$.*

*Proof.* All previously undefined terms and terminology in this proof are from [6]. We show that the graph $\mathbb{R}^k(\lambda)$ is a $k$-separator, and hence by Lemma 2.7 [6] is in $\mathscr{A}\mathscr{R}_k$. Let $D = [d_{ij}]$ be the $k \times k$ matrix with entries $d_{ij} = 2\lambda{+}2$ if $i \neq j$ and $d_{ij} = 0$ otherwise. It is clearly a distance matrix. Let $L = \{l_1, \dots, l_k\}$ be a set of size $k$, and let $f(l_i) = 2\lambda{+}1$ for all $1 \leq i \leq k$. It is easy to verify that (i) $(f(l_1), \dots, f(l_k))$ does not dominate any member of $\mathcal{L}(K, L)$, and that (ii) $M(K, L) = 3\lambda + 3$. Hence a tuple $(x_1, \dots, x_k)$ is admissible if and only if (a) $x_i + x_j \geq d_{ij} = 2\lambda + 2$ for $i \neq j$, and (b) it is not dominated by $(2\lambda + 1, \dots, 2\lambda + 1)$, i.e. it has at least one entry $x_j \geq 2\lambda + 2$, and (c) it is dominated by $(3\lambda + 3, \dots, 3\lambda + 3)$, i.e. $x_i \leq 3\lambda + 3$ for all $i$. Hence $\mathbb{R}^k(\lambda) = R(D, L, f) \in \mathscr{A}\mathscr{R}_k$.

□

**Lemma 5.14.** *Let $k \geq 2$. The class $\mathscr{A}\mathscr{R}_k$ is closed under products and retracts.*

*Proof.* This was first proved in [14] (see also [18]) but we can give a simple proof using Theorem 5.7. Suppose that $\mathbb{R}$ is a retract of $\mathbb{H}$ and let $\mathcal{D}$ be a duality for $\mathbb{H}^c$ that consists of polyads. It is immediate that a $\tau_{\mathbb{R}}$-structure $\mathbb{G}$ maps to $\mathbb{R}$ if and only if it maps to $\mathbb{G}$ when viewed as a $\tau_{\mathbb{H}}$-structure. Hence the set of polyads in $\mathcal{D}$ that have only colours from $\mathbb{R}$ is a duality for $\mathbb{R}^c$. Now suppose that ${\mathbb{H}_1}^c$ and ${\mathbb{H}_2}^c$ both have polyad duality, say with sets of polyads $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. Construct a set of $\tau_{\mathbb{H}_1 \times \mathbb{H}_2}$-polyads $\mathcal{D}$ as follows: for each $k$-coloured polyad $\mathbb{T}$ in $\mathcal{D}_1$, with vertices $x_1, \dots, x_k$ coloured by $c_1, \dots, c_k$, and for each sequence $\alpha = (a_1, \dots, a_k) \in (H_2)^k$, construct the polyad $\mathbb{T}(\alpha)$ with same underlying graph as $\mathbb{T}$ but now $x_i$ is coloured by $(c_i, a_i)$. Proceed similarly with the polyads from $\mathcal{D}_2$. It is easy to verify that $\mathcal{D}$ is a duality for $(\mathbb{H}_1 \times \mathbb{H}_2)^c$. □

**Theorem 5.15.** *Let $\mathbb{H}$ be a connected reflexive graph and let $k \geq 2$. Then the following are equivalent:*

(1) $\mathbb{H} \in \mathscr{A}\mathscr{R}_k$;
(2) *there exist $\lambda, s \geq 1$ such that $\mathbb{H} \trianglelefteq (\mathbb{D}^k(\lambda))^s$.*

*Proof.* Immediate by Lemmas 5.10, 5.12, 5.13 and 5.14. □

The next slightly technical lemma essentially states that reflexive duals of non-polyad trees do not have polyad duality.

**Lemma 5.16.** *Let $\mathbb{T}$ be a tree which is not a polyad and let $\mathbb{K} = D_r(\mathbb{T})$. There exists a $\tau_{\mathbb{K}}$-tree $\mathbb{S}$ such that*

(1) *$(\mathbb{S}^\tau)^u$ is isomorphic to $\mathbb{T}$;*
(2) *$\mathbb{S}$ is a critical obstruction of $\mathbb{K}^c$;*
(3) *if $\mathbb{P}$ is a $\tau_{\mathbb{K}}$-polyad and $\mathbb{P} \to \mathbb{S}$ then $\mathbb{P} \to \mathbb{K}^c$.*

*Proof.* By the remark following Theorem 3.6 and because it is not a polyad, we may assume that $\mathbb{T}$ is an elementary $\tau_{\mathbb{H}}$-tree for some graph $\mathbb{H}$[1]; we may also assume that the unary relations on its leaves are all distinct, i.e. if $u$ and $v$ are distinct leaves of $\mathbb{T}$ with colours $h$ and $h'$ respectively then $h \neq h'$. For each non-leaf $v$ of $\mathbb{T}$, define a $\tau_{\mathbb{H}}$-polyad $\mathbb{P}_v$ as follows: it has a central vertex $v'$, and for each leaf $u$ of $\mathbb{T}$ coloured by $h$ it has a leaf $u'$ coloured by $h$, connected to $v'$ by a branch of length $d_{\mathbb{T}}(u,v)$, isomorphic to the unique path in $\mathbb{T}$ from $u$ to $v$. For every leaf $u$ of $\mathbb{T}$, let $\mathbb{T}_u$ be the structure obtained from $\mathbb{T}$ by deleting the colour on the vertex $u$.

Build a $\tau_{\mathbb{H}}$-structure $\mathbb{Y}$ as follows: it is obtained from the disjoint union of all the polyads $\mathbb{P}_v$ and the trees $\mathbb{T}_u$ by identifying vertices carrying the same colour; furthermore add all loops, and make every edge symmetric.

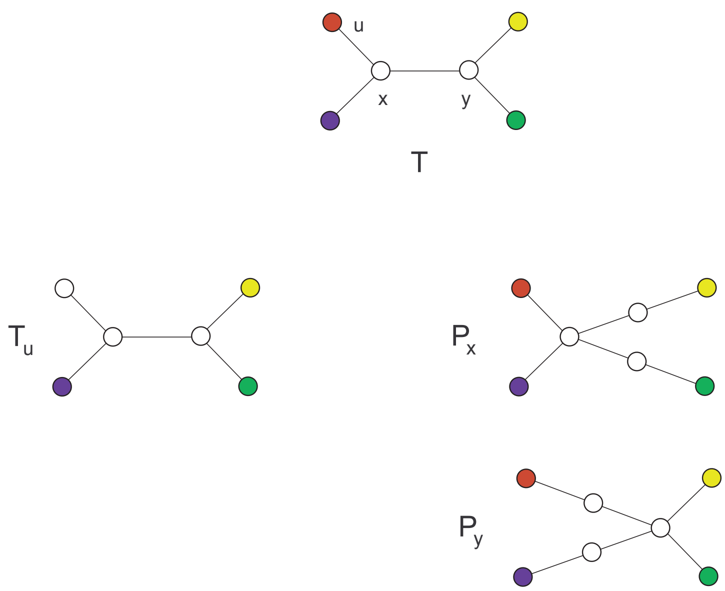


FIGURE 9. Some parts of the structure $\mathbb{Y}$ built from the smallest non-polyad tree $\mathbb{T}$, before glueing.

**Claim 1.** Let $v$ and $w$ be coloured vertices of $\mathbb{T}$ and let $v'$ and $w'$ be coloured vertices of $\mathbb{Y}$ such that $v$ and $v'$ have the same colour and $w$ and $w'$ have the same colour. Then $d_{\mathbb{Y}}(v',w') = d_{\mathbb{T}^u}(v,w)$; furthermore, if a path from $v'$ to $w'$ has length $d_{\mathbb{Y}}(v',w')$ then $v'$ and $w'$ are the only coloured vertices on it.

*Proof of Claim 1.* Consider any path in $\mathbb{Y}$ from $v'$ to $w'$ that passes through no other coloured element: it must lie entirely in some $\mathbb{T}_u$ or some $\mathbb{P}_z$. In the first case it is clear that the path will have length at least $d_{\mathbb{T}}(v,w)$ since $\mathbb{T}_u$ is an induced subtree of $\mathbb{T}$; in the second case we get the same result by noticing that the distance from $v'$ to $w'$ in $\mathbb{P}_z$ is at least as large as $d_{\mathbb{T}}(v,w)$. Since this must hold for all pairs of coloured vertices, if we have a path from $v'$ to $w'$ that contains some coloured vertex $z'$ where $z$ is a leaf of $\mathbb{T}$, then this path will have length at least $d_{\mathbb{T}^u}(v,z)+d_{\mathbb{T}^u}(v,z)$.

[1]In fact the orientation of the edges of $\mathbb{T}$ plays no real role in the proof.

Since $z$ is a leaf of $\mathbb{T}$ it cannot lie on a shortest path from $v$ to $w$ so this is in fact strictly greater than $d_{\mathbb{T}}(v,w)$, proving the second statement of our claim. Repeating the argument for an arbitrary number of coloured vertices on the path and using the triangle inequality shows that $d_{\mathbb{Y}}(v',w') \geq d_{\mathbb{T}}(v,w)$. For the reverse inequality, notice that $\mathbb{T}$ has more than 2 leaves since it is not a polyad. Let $u$ be a leaf of $\mathbb{T}$ distinct from $v$ and $w$ and let $u'$ denote the vertex of $\mathbb{Y}$ with the same colour as $u$. Then there is a path of length $d_{\mathbb{T}}(v,w)$ from $v'$ to $w'$ in $\mathbb{Y}$ passing through $\mathbb{T}_u$.

**Claim 2.** There exists a homomorphism $\phi : \mathbb{Y} \to D_r(\mathbb{T})$.

*Proof of Claim 2.* Since $\mathbb{Y}$ is reflexive and symmetric, it suffices to prove that $\mathbb{Y}$ admits a homomorphism to $D(\mathbb{T})$, in other words, that $\mathbb{T} \not\to \mathbb{Y}$. Suppose for a contradiction that there is a homomorphism $f : \mathbb{T} \to \mathbb{Y}$. Let $v$ and $w$ be distinct leaves of $\mathbb{T}$, and let $L$ denote the unique path from $v$ to $w$ in $\mathbb{T}$; this path has length at least 2 because $\mathbb{T}$ is not a polyad. By Claim 1 and because $f$ is non-expanding, $f(L)$ is a path of the same length in $\mathbb{Y}$ from $f(v) = v'$ to $f(w) = w'$, and hence it lies entirely in some $\mathbb{T}_u$ or some $\mathbb{P}_z$, call it $\mathbb{X}$. Let $\alpha$ be the unique neighbour of $u$ in $\mathbb{T}$. The unique path from $u$ to any other leaf of $\mathbb{T}$ passes through $\alpha$, and hence its image must lie entirely in $\mathbb{X}$. It follows that $f(\mathbb{T}) \subseteq \mathbb{X}$. Since $\mathbb{T}_u$ has no vertex with $u$'s colour, we must have $\mathbb{X} = \mathbb{P}_z$ for some $z$; it is clear that there is a homomorphism $g : \mathbb{P}_z \to \mathbb{T}$, and that $g \circ f : \mathbb{T} \to \mathbb{T}$ is the identity (since it fixes every leaf of $\mathbb{T}$), hence $\mathbb{T} = \mathbb{P}_z$, contradicting the fact that $\mathbb{T}$ is not a polyad.

We can now define the $\tau_{\mathbb{K}}$-tree $\mathbb{S}$ as follows: its underlying graph is $\mathbb{T}^\tau$; if $u$ is a leaf of $\mathbb{T}$ coloured $h$, let $v_h$ be the unique vertex in $\mathbb{Y}$ coloured $h$; we let the vertex $u$ of $\mathbb{S}$ have colour $\phi(v_h)$. Property (1) is immediate by construction of $\mathbb{S}$. We proceed to prove properties (2) and (3).

**Claim 3.** (i) Let $\mathbb{P}$ be a non-degenerate $\tau_{\mathbb{H}}$-polyad such that $\mathbb{P} \to \mathbb{T}$. Then there exists a vertex $v$ of $\mathbb{T}$ such that $\mathbb{P} \to \mathbb{P}_v$. (ii) Let $\mathbb{T}'$ be a proper connected substructure of $\mathbb{T}$. Then there exists a leaf $u$ of $\mathbb{T}$ such that $\mathbb{T}' \to \mathbb{T}_u$.

*Proof of Claim 3.* (i) Let $f : \mathbb{P} \to \mathbb{T}$ and let $c$ denote the center of $\mathbb{P}$. If $f(c) = v$ is not a leaf of $\mathbb{T}$ it is easy to see that $\mathbb{P}$ admits a homomorphism to $\mathbb{P}_v$. Now suppose that $f(c) = u$ where $u$ is a leaf of $\mathbb{T}$. Then all neighbours of $c$ in $\mathbb{P}$ are mapped to the unique neighbour of $u$ in $\mathbb{T}$, call it $v$. It follows that $\mathbb{P}$ admits a homomorphism to $S_v$. (ii) Since $\mathbb{T}'$ is connected and different from $\mathbb{T}$ it means that some colour $h$ appearing in $\mathbb{T}$ does not appear in $\mathbb{T}'$, so $\mathbb{T}' \to \mathbb{T}_u$ where $u$ is the vertex of $\mathbb{T}$ with colour $h$.

To prove (2), suppose for a contradiction that there exists a homomorphism $\psi : \mathbb{S} \to D_r(\mathbb{T})$. Let $u$ be a leaf of $\mathbb{T}$ which has colour $h$, so that $u$ has colour $\phi(v_h)$ in $\mathbb{S}$. Then $\psi(u) = \phi(v_h)$ since $\psi$ preserves colours, and since $\phi(u)$ has colour $h$, $\psi$ actually defines a homomorphism from $\mathbb{T}$ to $D_r(\mathbb{T})$ (as $\tau_{\mathbb{H}}$-structures), a contradiction, hence $\mathbb{S}$ is an obstruction. To see that it is critical, let $\mathbb{S}'$ be a proper substructure of $\mathbb{S}$. Consider the $\tau_{\mathbb{H}}$-structure $\mathbb{T}'$ obtained from $\mathbb{S}'$ in the obvious way, namely, its underlying digraph is $(\mathbb{S}')^\tau$ and vertices that are coloured in $\mathbb{S}'$ get the colour they have in $\mathbb{T}$. Obviously $\mathbb{T}'$ is a proper substructure of $\mathbb{T}$. By Claim 3, for each connected component $\mathbb{C}$ of $\mathbb{T}'$ there exists a leaf $u$ of $\mathbb{T}$ such that $\mathbb{C} \to \mathbb{T}_u$, and hence there exists a homomorphism $\gamma : \mathbb{T}' \to \mathbb{Y}$. It is clear that this map induces a homomorphism (of $\tau_{\mathbb{K}}$-structures) from $\mathbb{S}'$ to $D_r(\mathbb{T})$.

Finally we prove (3): let $\mathbb{P}$ be a $\tau_{\mathbb{K}}$-polyad such that $\delta : \mathbb{P} \to \mathbb{S}$. Notice that (i) the only colours that can appear on vertices of $\mathbb{P}$ have to be colours in $\mathbb{S}$, namely, of the form $\phi(v_h)$ for some colour $h$ appearing in $\mathbb{T}$, and (ii) $\mathbb{P}$ must be non-degenerate, since $\mathbb{S}$ is not a polyad. Let $\mathbb{P}'$ be the (non-degenerate) $\tau_{\mathbb{H}}$-polyad obtained from $\mathbb{P}$ as follows: the underlying digraph is the same as $\mathbb{P}$'s, and if $u$ in $\mathbb{P}$ has colour $\phi(v_h)$, then its colour in $\mathbb{P}'$ will be $h$. By definition of $\mathbb{S}$, the map $\delta$ now defines a homomorphism from $\mathbb{P}'$ to $\mathbb{T}$. By Claim 3, there exists some vertex $v$ of $\mathbb{T}$ such that $\mathbb{P}' \to \mathbb{T}_v$, and hence $\mathbb{P}' \to \mathbb{Y}$; this induces a homomorphism from $\mathbb{P}$ to $D_r(\mathbb{T})$ and we are done. □

Let $\mathscr{N}\mathscr{U}_k$ denote the class of graphs that admit a $k$-ary NU polymorphism. The first two results were previously known [23], see also [5], [6].

**Theorem 5.17.** *Let $k \geq 2$.*

(1) $\mathscr{A}\mathscr{R}_k \subseteq \mathscr{N}\mathscr{U}_{k+1}$;
(2) $\mathscr{A}\mathscr{R}_2 = \mathscr{N}\mathscr{U}_3$;
(3) $\mathscr{A}\mathscr{R}_3 = \mathscr{N}\mathscr{U}_4$;
(4) *If $k > 3$ and $\mathbb{T}$ is a tree with $k$ leaves which is not a polyad, then $D_r(\mathbb{T}) \in \mathscr{N}\mathscr{U}_{k+1} \setminus \mathscr{N}\mathscr{U}_k$ and $D_r(\mathbb{T})^c \notin \mathscr{A}\mathscr{R}$.*

*Proof.* Statements (1), (2) and (3) are immediate by Theorems 3.9 and 5.7 and the fact that elementary trees with at most 3 coloured elements are polyads. To prove (4), notice that $D_r(\mathbb{T})$ admits a critical obstruction with $k$ leaves by Lemma 5.16, and hence it does not admit a $k$-ary NU polymorphism; on the other hand by Lemma 4.3 and Corollary 4.4 it admits one of arity $k+1$. Finally, it is immediate by Lemma 5.16 (2) and (3) that $D_r(\mathbb{T})^c$ cannot have polyad duality and hence is not an absolute retract by Theorem 5.7.

□

The following result was the prototype/motivation behind our investigations we stated in the introduction:

**Corollary 5.18.** [19] *Let $\mathbb{H}$ be a reflexive graph. Then the following conditions are equivalent:*

(1) $\mathbb{H} \in \mathscr{N}\mathscr{U}_3$;
(2) $\mathbb{H} \in \mathscr{A}\mathscr{R}_2$;
(3) $\mathbb{H}$ *is a retract of a product of paths.*

*Proof.* Immediate by Lemma 5.9 (2) and Theorems 5.15 and 5.17. □

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