Matrix Partitions of Perfect Graphs

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Abstract

Given a symmetric m by m matrix M over $0, 1, *$, the M-partition problem asks whether or not an input graph G can be partitioned into m parts corresponding to the rows (and columns) of M so that two distinct vertices from parts i and j (possibly with $i = j$) are non-adjacent if $M(i, j) = 0$, and adjacent if $M(i, j) = 1$. These matrix partition problems generalize graph colourings and homomorphisms, and arise frequently in the study of perfect graphs; example problems include split graphs, clique and skew cutsets, homogeneous sets, and joins.

In this paper we study M-partitions restricted to perfect graphs. We identify a natural class of 'normal' matrices M for which M-partitionability of perfect graphs can be characterized by a finite family of forbidden induced subgraphs (and hence admits polynomial time algorithms for perfect graphs). We further classify normal matrices into two classes: For the first class, the size of the forbidden subgraphs is linear in the size of M; for the second class we only prove exponential bounds on the size of forbidden subgraphs. (We exhibit normal matrices of the second class for which linear bounds do not hold.)

We present evidence that matrices M which are not normal yield badly behaved M-partition problems: There are polynomial time solvable M-partition problems that don't have finite forbidden subgraph characterizations for perfect graphs. There are M-partition problems that are NP-complete for perfect graphs. There are classes of matrices M for which even proving 'dichotomy' of the corresponding M-partition problems for perfect graphs – i.e., proving that these problems are all polynomial or NP-complete – is likely to be difficult.

1 Introduction

The *M-partition problem* was introduced in [10]: Let M be a fixed symmetric $m \times m$ matrix with entries $M_{i,j} \in \{0, 1, *\}$. An M-partition of a graph G is a partition of the vertices of G into m *parts*, indexed by the rows (and columns) of the matrix M, such that for distinct vertices x and y of the graph G, placed in parts i and j (possibly with $i = j$) respectively, we have the following:

- if $M(i, j) = 0$, then xy is not an edge of G;
- if $M(i, j) = 1$, then xy is an edge of G.

(If $M(i, j) = *$, then xy may or may not be an edge in G.)

An instance of the M-partition problem is a graph G, and a solution to the instance is an M-partition of G.

Each matrix M defines a complementary matrix $\overline{M}$, obtained by exchanging all 0's and 1's. It is easy to see that G admits an M-partition if and only if $\overline{G}$ admits an $\overline{M}$-partition.

Suppose H is a graph with m vertices and M is obtained from the adjacency matrix of H by replacing each 1 by $*$. Then each homomorphism (edge-preserving vertex mapping) f of G to H corresponds to an M-partition of G, where the parts are $f^{-1}(h), h \in V(H)$. In particular, when $H = K_m$, the matrix M is the matrix with all diagonal entries 0 and all off-diagonal entries $*$, and an M-partition of G is simply an m-colouring of G.

Matrix partitions not only generalize colourings and homomorphisms, but also unify many partition problems arising in the study of perfect graphs. Often these problems are not stated in terms of partitions, but are in fact equivalent to partition problems. For instance, it is evident that G is a *split graph* (admits a partition into an independent set and a clique [13]), if and only if it admits an M-partition where M is the matrix

$$\begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}.$$

It is only slightly less evident that a graph G has a clique cutset [21, 22], if it admits an M-partition *into non-empty parts* where M is the matrix

$$\begin{pmatrix} 1 & * & * \\ * & * & 0 \\ * & 0 & * \end{pmatrix}.$$

A similar approach allows us to model by M-partitions problems such as having a skew cutset [6, 7], a homogeneous set [19], or being a join of various kinds [5]. These connections are explored in more detail in [10], where it is in particular explained how to model restrictions on the size of the parts (for instance requiring some parts to be non-empty, or to have at least a certain number of elements) by introducing lists.

In [10] we gave polynomial time algorithms for many list M-partition problems, and quasi-polynomial ($n^{O(\log n)}$) time algorithms for certain others. In [9] we have shown that all list M-partition problems are solvable in quasi-polynomial time, or are NP-complete. Many of our quasi-polynomial time bounds from [10] were improved to polynomial time bounds in [4, 7], but it is not known whether or not *all* the quasi-polynomial time list M-partition problems are polynomial time solvable (even for matrices of size four [4]). Thus it is an open question whether or not list M-partition problems enjoy *dichotomy*, i.e., whether or not all list M-partition problems are polynomial time solvable or NP-complete. Ladner [17] proved that unless $P = NP$, there are in NP problems that are neither in P nor NP-complete. (Of course, no such problems are known, since this would imply that $P \neq NP$.) The dichotomy of *constraint satisfaction problems* [12] is an important open question in theoretical computer science. Without defining constraint satisfaction problems explicitely, we note that constraint satisfaction problems include those M-partition problems in which M has no 1 (or, by complementation, no 0), i.e., is obtained from the adjacency matrix of a graph H by replacing each 1 with an $*$ as explained above. Dichotomy for these M-partition problems (known as *H-colouring problems*) has been proved by Hell and Nesetril [16]. However, dichotomy is not known for M-partition problems (H-colouring problems) for digraphs, i.e., for $0, *$-matrices M that are not necessarily symmetric. In fact, Feder and Vardi have shown [12] that dichotomy for H-colouring problems for digraphs would imply dichotomy for all constraint satisfaction problems. For general matrices (those having 0's, 1's, and $*$'s), it is no longer the case that M-partition problems are constraint satisfaction problems, so it is in principle possible that dichotomy holds for constraint satisfaction problems but not for M-partition problems [9].

In this paper, we consider the restriction of M-partition problems (without lists) to perfect graphs. A graph G is *perfect* [1] if it and all of its induced subgraphs have chromatic number equal to its maximum clique size. This also implies that G and all of its induced subgraphs have the chromatic number of their complement equal to maximum size of their independent set [18].

If M has a diagonal $*$, then every graph has an M-partition, since all vertices can be placed in the corresponding (unrestricted) part. Thus we shall focus on matrices M with $0,1$-diagonals; in fact, we may take M to consist of two diagonal block matrices A, B, where A is a k by k matrix with zero diagonal, B is an ℓ by ℓ matrix with a diagonal of ones, and a k by ℓ off-diagonal matrix C (and its ℓ by k transpose). Note that $m = k + \ell$. We shall categorize the matrices M according to what kind of entries they have in the blocks A, B, C: Let $\mathcal{A}$ denote the set of entries ($0, 1$, or $*$) which appear in the off-diagonal positions of A, let $\mathcal{B}$ be the set of all entries which appear in the off-diagonal positions of B, and let $\mathcal{C}$ be the set of entries which appear in C. We shall say that a subset of $\{0, 1, *\}$ is *normal* if it does not contain both an $*$ and another element. Thus every normal set is either $\{*\}$ or a subset of $\{0, 1\}$. We shall say that a matrix M is *normal* if all of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are normal sets. Note that, in particular, any matrix in which all $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are singleton sets, is normal.

The main result of this paper is the fact that for all normal matrices M there is characterization of M-partitionable perfect graphs by a finite set of induced forbidden subgraphs. (Thus there is also a polynomial time algorithm to solve the corresponding M-partition problem restricted to perfect graphs.)

We phrase our results in terms of minimal obstructions. Given a matrix M, a graph G is a *minimal obstruction* for M-partitionability, if G has no M-partition, but each proper induced subgraph of G has an M-partition. If we show, for a matrix M, that all minimal obstructions that are perfect have at most a certain number $f(M)$ of vertices, then M-partitionability of perfect graphs can be characterized by a finite set of forbidden induced subgraphs, and the M-partition problem for perfect graphs is polynomial time solvable.

We further classify normal matrices into two classes:

We shall say that M is *of class one* if either M contains no $*$, i.e, if $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are all subsets of $\{0, 1\}$, or if $\mathcal{A}$ and $\mathcal{B}$ are any normal sets, and $\mathcal{C}$ is $\{0\}$ or $\{1\}$. We shall prove that for each normal matrix M of class one, the size of minimal obstructions that are perfect is at most $O(k\ell)$. (Note that $k\ell$ is the number of entries of C, which is smaller than the number of entries of M; thus we view this bound as being linear in the size of M.)

We shall say that a normal matrix M is *of class two* if it is not of class one, i.e., if either $\mathcal{C} = \{*\}$ and $\mathcal{A}$ and $\mathcal{B}$ are any normal sets, or $\mathcal{C} = \{0, 1\}$, and one of the normal sets $\mathcal{A}$, $\mathcal{B}$ is $\{*\}$. We shall prove that for each class two matrix M the size of minimal obstructions that are perfect is still bounded by a function of M. However, in this case the bound we give (assuming

$k \leq \ell$, by complementation if necessary) is only $O(\ell)^{O(k\ell)}$.

Let $R_{k,\ell}$ denote the disjoint union of $\ell+1$ cliques of size $k+1$. The result of [15] states that when $\mathcal{A} = \mathcal{B} = \mathcal{C} = \{*\}$, i.e., for partitions into k cliques and ℓ independent sets, there is only one minimal obstruction that is chordal, namely $R_{k,\ell}$. Thus we have situations in which minimal obstructions of size $(k+1)(\ell+1) = O(k\ell)$ are possible. It is interesting to observe that for perfect graphs, in this case, we must admit minimal obstructions with more than $(k+1)(\ell+1)$ vertices: For instance, in the case $k = 2$, $\ell = 1$, (and $\mathcal{A} = \mathcal{B} = \mathcal{C} = \{*\}$), the (bipartite, hence perfect) graph H, obtained from the three-dimensional cube by deleting one vertex, has seven (more than $(k+1)(\ell+1) = 6$) vertices and is a minimal obstruction for partitions into two cliques and an independent set. Nevertheless, the size $(k+1)(\ell+1)$ seems to be the right dividing line between small and large obstructions, as we detail below. In the last section, we shall also describe example normal matrices M of the second class, for which there is no $O(k\ell)$ bound on the size of minimal obstructions. These obstructions will be trees, thus perfect.

We now argue that for matrices M that are not normal, the M-partition problems are more difficult.

In [11] we have constructed matrices M which yield NP-complete M-partition problems, even when restricted to split, and hence perfect, graphs. These matrices have $\mathcal{A} = \{0\}$, $\mathcal{B} = \{*\}$, and $\mathcal{C} = \{0, *\}$, and thus are not normal. For these matrices M we do not expect polynomial time M-partition algorithms, and hence expect infinitely many minimal obstructions.

Even in cases when the M-partition problem has a polynomial time algorithm, there may be infinitely many perfect minimal obstructions. For instance, consider the matrix M

$$\begin{pmatrix} 0 & 1 & * \\ 1 & 0 & * \\ * & * & 0 \end{pmatrix},$$

in which $\mathcal{A} = \{1, *\}$, and hence M is not normal. It has been shown in [2, 10] that this M-partition problem can be solved in polynomial time. Yet there are infinitely many perfect minimal obstructions: Let G_t be the path with vertices $1, \ldots, 2t$ with an additional vertex 0 and edges $0i$ for $i = 2, 3, \ldots, 2t-1$. Then each G_t is chordal, and hence perfect, and does not admit an M-partition. Clearly, no G_t is an induced subgraph of a G_s, unless $t = s$, so the number of minimal obstructions is infinite.

In the last section, we shall offer evidence that for matrices M with $\mathcal{C} = \{0, 1, *\}$ (hence not normal) it will probably be difficult even to prove dichotomy of M-partition problems for prefect graphs, i.e., to prove that all such M-partition problems are polynomial or NP-complete, when restricted to perfect graphs. Specifically, we shall sketch a proof showing that dichotomy for these problems would imply dichotomy for all constraint satisfaction problems, which, as we have suggested above, is one of the big open questions in theoretical computer science. This situation arises even if $\mathcal{A} = \{0, *\}$ and $\mathcal{B} = \{0\}$, as long as $\mathcal{C} = \{0, 1, *\}$; it also arises when $\mathcal{A} = \{0, 1, *\}$, $\mathcal{B} = \{*\}$, and $\mathcal{C} = \{0, 1\}$.

Each M-partition of a graph G partitions the vertices of G into k independent sets, corresponding to the parts in A, and ℓ cliques, corresponding to the parts in B (with some additional restrictions). We denote by G_A the subgraph induced by the union of the k independent sets, and by G_B the subgraph induced by the union of the ℓ cliques. Note that G_A is k-colourable, and that the complement of G_B is ℓ-colourable.

In [11] we consider similar problems in the context of chordal graphs. The bounds are generally lower, and the algorithms are more efficient.

2 Class One Matrices

In this section we focus on normal matrices M of class one, and show that M-partitionability of perfect graphs can be characterized by minimal obstructions with at most $(k+1)(\ell+1)$ vertices.

In [8] we have studied general M-partition problems (not restricted to perfect graphs), for the class of matrices M which do not have $*$'s, i.e., matrices in which $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are all subsets of $\{0, 1\}$. These M-partition problems have only small minimal obstructions (for all graphs, not just for perfect graphs). The relevant result for our purposes is the following theorem from [8]:

Theorem 2.1 *Let M be a matrix with $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ all subsets of $\{0, 1\}$. Then every minimal obstruction to M-partitionability has at most $(k+1)(\ell+1)$ vertices.*

Moreover, if both $k \geq 1$ and $\ell \geq 1$, then there is at most one minimal obstruction with exactly $(k+1)(\ell+1)$ vertices.

Given a graph G, we say that vertices u and v are *similar* if u and v are not adjacent, and have exactly the same set of neighbours. A graph without two similar vertices is called *point determining* [20].

We now describe which matrices M without $*$'s do admit a minimal obstruction with $(k+1)(\ell+1)$ vertices, and what the obstruction looks like

[8]. Suppose L_k is a vertex transitive point determining graph with $k+1$ vertices. Let $L_{k,\ell}$ be obtained from L_k by replacing each vertex with $\ell+1$ independent vertices, and making two replacing vertices adjacent in $L_{k,\ell}$ if and only if the vertices they replaced were adjacent in L_k. Let M' be the adjacency matrix of L_k. Finally, let M be obtained from M' by replacing $M'(k+1,k+1)=0$ with the ℓ by ℓ identity matrix, and setting all $M(i,j)$ with $i>k, j\leq k$ equal to $M'(k+1,j)$, and all $M(i,j)$ with $i\leq k, j>k$ equal to $M'(i,k+1)$.

If a matrix M or its complement is obtained this way from some L_k, then it has the unique minimal obstruction with $(k+1)(\ell+1)$ vertices, namely $L_{k,\ell}$ or $\overline{L_{k,\ell}}$ respectively; otherwise all minimal obstructions for M have fewer than $(k+1)(\ell+1)$ vertices.

All other cases of normal matrices of class one are covered by the following theorem:

Theorem 2.2 *If M is a normal matrix in which $\mathcal{C}$ is $\{0\}$ or $\{1\}$, then every perfect graph which is a minimal obstruction has at most $(k+1)(\ell+1)$ vertices.*

Moreover, if $k\geq 1$ and $\ell\geq 1$, then in the case $\mathcal{C}=\{0\}$ the only minimal obstruction with $(k+1)(\ell+1)$ vertices is $R_{k,\ell}$ (and in the case $\mathcal{C}=\{1\}$, the complement of $R_{\ell,k}$), and it is a minimal obstruction only if both $\mathcal{A}$ and $\mathcal{B}$ are singletons, not equal to $\{0\}$ and $\{1\}$ respectively.

Both theorems exclude the claim on uniquess of minimal obstructions with $(k+1)(\ell+1)$ vertices in the case k or ℓ is zero. In this case there are, up to complementation, just two cases of normal matrices: We may assume that $\ell=0$, i.e., that M has zero diagonal, and that either all off-diagonal entries of M are $*$, or none is $*$. In the first case, we do have the uniqueness of minimal obstructions with $(k+1)(\ell+1)$ vertices, Lemma 2.3. In the second case, there exist matrices M for which the minimal obstructions with $(k+1)(\ell+1)$ vertices are not unique; however, we have shown that there never are more that two minimal obstructions [8].

A careful reading of the proofs below will show that the bound applies more generally. Instead of perfectness of the minimal obstruction G, we only need to assume, if $\mathcal{A}=\{*\}$, that every induced subgraph of G which does not have a clique on $k+1$ vertices is k-colourable, and, if $\mathcal{B}=\{*\}$, that every induced subgraph of G which does not have an independent set on $\ell+1$ vertices has a complement which is ℓ-colourable. (Thus if neither $\mathcal{A}$ nor $\mathcal{B}$ is $\{*\}$, no restriction is needed.)

To prove the theorem, we shall asume that $\mathcal{C} = \{0\}$; the other case follows by complementation. If $\mathcal{A}$ and $\mathcal{B}$ are both subsets of $\{0, 1\}$, we appeal to the previous theorem; if M or $\overline{M}$ is associated with some L_k or $\overline{L_k}$ respectively, then the fact that L_k is vertex transitive and the fact that $\mathcal{C} = \{0\}$ imply that L_k is an independent set and $\overline{L_k}$ a clique, and hence the unique obstruction $L_{k,\ell}$ or $\overline{L_{k,\ell}}$ is equal to $R_{k,\ell}$ or its complement. Otherwise, one of the following three lemmas applies.

Lemma 2.3 *Assume that $\mathcal{A} = \mathcal{B} = \{*\}$ and $\mathcal{C} = \{0\}$.*

Every perfect graph which is a minimal obstruction is either isomorphic to $R_{k,\ell}$ or has strictly fewer than $(k+1)(\ell+1)$ vertices.

Proof. Let G be a perfect graph. We want to partition G into a k-colourable induced subgraph G_A and an induced subgraph G_B whose complement is ℓ-colourable, with no edges joining G_A and G_B. Thus each component of G must be included entirely either in G_A or in G_B. All components C of G that contain a clique of size $k+1$ must be included in G_B. Let D be the union of all components C of G that contain a clique of size $k+1$. Then G is partitionable if and only if D does not have an independent set of size $\ell + 1$. Indeed, if D has $\ell + 1$ independent vertices that its components cannot all be contained in G_B. On the other hand, if D does not have $\ell+1$ independent vertices then its complement can be ℓ-coloured, since G is perfect. In other words, then D can be covered by ℓ cliques. The subgraph $G - D$ can be k-coloured for the same reason; thus G can be partitioned into k independent sets and ℓ cliques.

If $k = 1$ then a minimal obstruction is $\ell + 1$ independent vertices v_i and a neighbor w_i for each in D, and thus has at most $2(\ell + 1)$ vertices. If a minimal obstruction has exactly $2(\ell+1)$ vertices, then no v_i is adjacent to w_j for $j \neq i$, otherwise we can remove w_i. Furthermore, no two distinct w_i, w_j are adjacent, otherwise we can remove v_i and use w_i as an independent vertex instead of v_i, so D is $R_{k,\ell}$.

Now we assume $k \geq 2$ and proceed by induction on ℓ. We know that every component U of D contains a $(k+1)$-clique $K(U)$. Let S be an independent set in D with $(\ell+1)$ vertices. By the minimality of G, we have the following property:

(*) *For every U, no proper connected induced subgraph of U contains a clique of size $k+1$ and an independent set of size $|S \cap U|$.*

Now consider a specific component W. If $W = K(W)$ then we can delete it and proceed by induction. (The number of cliques needed to cover the components of D is reduced by one). Otherwise, we see that we can choose a spanning tree of W which has a leaf v outside $K(W)$, for instance by contracting K to a vertex. But now, by the minimality of G, the graph $G - v$ can be covered by ℓ cliques, so G can be covered by $\ell + 1$ cliques. We fix such a set of $\ell + 1$ cliques.

We let $F(U)$ be the subfamily of these cliques which intersects U and write $n(U) = |F(U)|$. The remarks of the last paragraph imply that we can choose our cliques so that each $F(U)$ contains a singleton clique disjoint from $K(U)$. We let $S(U) = |S \cap U|$ and note that $|S(U)| = n(U)$, and that each clique of $F(U)$ contains exactly one vertex of $S(U)$. So, our singleton clique is a vertex of $S(U)$.

The graph whose vertices are the cliques of $F(U)$ and in which two are joined if there is an edge between them is obviously connected, and has at least two vertices. The edges of a spanning tree for this graph correspond to a tree $T(U)$ of U which contains at most $2n(U)-2$ vertices and intersects every clique of $F(U)$. Consider the graph U' obtained from $T(U)$ by the addition of the at most $n(U) - 1$ other vertices of $S(U)$. This U' is a connected subgraph of U with the same maximum independent set size $n(U)$, and at most $3n(U) - 3$ vertices.

We may assume $T(U)$ contains at least one vertex of $K(U)$, because if $K(U)$ meets two cliques of $F(U)$ then we can join them with an edge of $K(U)$, and if $K(U)$ is contained in a clique of $F(U)$ then $K(U)$ can be chosen in this clique of $F(U)$ to contain a chosen vertex. Adding to U' the k other vertices of $K(U)$, we see that we have a connected subgraph of U which by $(*)$ must be U and hence $|U| \leq 3n(U) - 3 + k$. Thus, if r is the number of components of G, then $|D| \leq (k-3)r + 3(\ell+1) < (k+1)(\ell+1)$. $\square$

Lemma 2.4 *Assume that $k > 0$, and $\mathcal{A} = \{*\}$, $\mathcal{B} \subseteq \{0, 1\}$, and $\mathcal{C} = \{0\}$.*

Every perfect graph which is a minimal obstruction is either isomorphic to $R_{k,\ell}$ or has strictly fewer than $(k+1)(\ell+1)$ vertices.

Proof. Each connected component that does not have a clique of size $k+1$ can be placed in G_A. Any minimal obstruction G is thus contained in the union D of the components that contain a clique of size $k+1$. If D contains $\ell+1$ independent vertices, then G does not even have an M-partition where M is the matrix with $\mathcal{A} = \mathcal{B} = \{*\}$ and $\mathcal{C} = \{0\}$, thus G is $R_{k,\ell}$ or has fewer than $(k+1)(\ell+1)$ vertices, by Lemma 2.3.

Otherwise D has a maximum independent set of size at most ℓ, and so G can be covered by ℓ cliques. The components of D, which must be placed in G_B, have an obstruction for G_B from Theorem 2.1 of size at most $\ell + 1$.

Now consider a specific component U of D with $K(U)$, $F(U)$, $S(U)$, $n(U) = |F(U)| = |S(U)|$, defined as in Lemma 2.3. The s components that have $n(U) = 1$ can be taken to be a clique of size $k+1$ containing all the vertices of the obstruction for G_B of size $k+1$ in U. The remaining t components have $n(U) \geq 2$. The graph whose vertices are the cliques of $F(U)$ and in which two are joined if there is an edge between them is obviously connected, and has at least two vertices. The edges of a spanning tree for this graph correspond to a tree $T(U)$ of U which contains at most $2n(U) - 2$ vertices and intersects every clique of $F(U)$.

Adding the k other vertices of $K(U)$ and the other $m(U)$ vertices of the obstruction for G_B of size $k+1$, we see that we have a connected subgraph of U which must be U and hence $|U| \leq 2n(U) - 2 + k + m(U)$. Thus, $|D| \leq (k+1)s + 2(\ell - s) + (k-2)t + (k+1-s) = (k-2)(s+t) + 2\ell + k + 1 \leq k\ell + k + 1$. This gives an obstruction with fewer than $(k+1)(\ell+1)$ vertices. $\square$

Lemma 2.5 *Assume that $\ell > 0$, and $\mathcal{A} \subseteq \{0,1\}$, $\mathcal{B} = \{*\}$, and $\mathcal{C} = \{0\}$.*

Every perfect graph which is a minimal obstruction is either isomorphic to $R_{k,\ell}$ or has strictly fewer than $(k+1)(\ell+1)$ vertices.

Proof. Let G be a minimal perfect obstruction. The case $k = 1$ is covered by Lemma 2.3, and the case $\ell = 1$ by Theorem 2.1. Thus we assume $k \geq 2, \ell \geq 2$.

Recall that two nonadjacent vertices u, v are similar if they have exactly the same set of neighbours. Similarity is easily shown to be an equivalence relation; it partitions the vertices of G into independent sets $S_1, \ldots, S_r$, with two sets either completely nonadjacent or completely adjacent. Since $\mathcal{A}$ has no $*$'s, similarity in any component that is contained in G_A has at most k equivalence classes. Thus, if a component U of G contains at least $k+1$ sets S_i, then it cannot go to G_A. Assume that U is such a component.

Suppose first that our component U has $\ell + 1$ independent vertices. Remove from U sufficiently many vertices (not cutpoints) until we are left with a connected subgraph U' of U with maximum independent set size $n(U') = \ell + 1$. At this point, we proceed as in Lemma 2.3, selecting a connected subgraph U'' of U' with the same maximum independent set size $n(U')$ and at most $3n(U') - 3 = 3\ell$ vertices. We also select a sequence of vertices $v_1, v_2, \ldots, v_r$ in U, each in a different S_i, so that for each $t \leq r$, the subgraph V_t of U induced by $v_1, \ldots, v_t$ has at most one pair of similar vertices. This can be done inductively, by starting with any adjacent v_1, v_2, and extending V_t to V_{t+1} by letting v_{t+1} be any vertex adjacent to exactly one of the two vertices similar in V_t (which are not similar in U since they come from different S_i's). If V_t has no two similar vertices, then any vertex adjacent to V_t can be taken as v_{t+1}. Then each V_t is connected, and either V_{k+1} or V_{k+2} is an obstruction for the matrix A. We are free to choose the first two vertices v_1, v_2 in U''; taking this obstruction together with U'' yields an obstruction W to our problem, of size at most $3n(U') - 3 + k = 3\ell + k < (k+1)(\ell+1)$.

Now assume that our component U has maximum independent set size $n(U) \leq \ell$. We induct on ℓ. If $n(U) = 1$, then U is a clique, and we let W be a clique with $k+1$ vertices from U. If $2 \leq n(U) \leq \ell$, then as in the previous case we obtain a smaller component W with at most $3n(U) - 3 + k < (k+1)n(U)$ vertices, that contains an obstruction for A, an independent set of size $n(U)$, and is connected. Inductively, we know that $G - U$ has an obstruction for the problem with ℓ replaced by $\ell' = \ell - n(U)$ with at most $(k+1)(\ell - n(U) + 1)$ vertices, which combined with W gives an obstruction of size at most $kn(U) + (k+1)(\ell - n(U) + 1) \leq (k+1)(\ell+1)$. Equality holds only if $n(U) = 1$, in which case we have inductively an $R_{k,\ell-1}$, giving us an $R_{k,\ell}$.

In the remaining cases each component U contains at most k sets S_i. Suppose first that each S_i has only one vertex. In that case G is point determining, and by a result in [20] contains an induced subgraph $G' = G - v_0$ with one less vertex that is also point determining. Since G is a minimal obstruction, G' has an M-partition. The subgraph G'_B that goes to the parts in B has maximum independent set size at most ℓ, and thus has at most ℓ components, each with at most k vertices, so G'_B has at most $k\ell$ vertices. The subgraph G'_A that goes to the parts in A has at most k vertices, since G'_A is point determining and so no two vertices of G'_A can go to the same part. Thus G' has at most $k\ell + k$ vertices, and G is an obstruction with at most $k\ell + k + 1 < (k+1)(\ell+1)$ vertices.

Suppose next that some S_i has at least two vertices. In this case we compute the following parameter t. Let t_1 denote the number of isolated vertices in G. Let t_2 denote the maximum, over all components C of G, of the number of vertices in C minus the number of sets S_i included in C. Let t be the larger of the numbers $t_1 - 1$ and t_2. Note that $t \geq 1$.

Let S be a maximum independent set in G. The set S is the union of sets S_i, and we may assume that every S_j not in S has exactly one element. Otherwise removing a vertex v from an S_j not in S with at least two elements still gives an obstruction, because if S_j goes to a part in A then we could add the removed vertex v to the same part in A, and if S_j goes to a part in B then adding the removed vertex v to G_B does not increase the size $|S|$ of a maximum independent set in B, and so there is still a solution in B.

We now define a vertex v_0 and a set C of vertices of G containing v_0. If $t = t_1 - 1$, let v_0 be an isolated vertex, and C the set of all isolated vertices. If $t = t_2$, let v_0 be a vertex in a set S_i that has at least two vertices, and in a component U of G in which t_2 is maximized; let $C = U$. If $t = 1$ then we choose the set C so that it contains the smallest number, u, of sets S_i. (Note that $u = 1$ if C is the set of isolated vertices.) Consider now an M-partition of $G' = G - v_0$. In this M-partition, the vertices in $C - v_0$ cannot go to G'_A, since then we could put v_0 in the same part as another vertex in the same S_i, so the vertices in $C - v_0$ go to G'_B. A component U in G'_B that has $n(U)$ vertices in S has a total of at most $n(U) + k - 1$ vertices, since sets S_i not in S have just a single vertex.

Suppose $t \geq 2$. Since each component U has at most $k - 1$ vertices not in S, we have that G'_B has at most $(k-1)(\ell - t + 1)$ vertices not in S, and so G'_B has at most $(k-1)(\ell - t + 1) + \ell$ vertices. A component C' in G'_A that contains u' sets S_i has at most $u' + t$ vertices, or at most $(u'+t)/u' = 1 + t/u'$ vertices per part of A. The number of vertices in G'_A is thus at most $k(1+t/2)$ plus an additional $t/2 - 1$ if there are isolated vertices in A. The total number of vertices in G is therefore at most $(k-1)(\ell - t + 1)$ $+\ell + k(1+t/2) + t/2 - 1 + 1 = k\ell + k + 2 - (t-2)(k-3)/2 \leq k\ell + k + \ell/2 + 1$ $< (k+1)(\ell+1)$.

Finally if $t = 1$, then G'_B has at most $k(\ell - 1)$ in components other than C, and at most u vertices in C, giving at most $k(\ell-1) + v$ vertices in G'_B. A component C' in G'_A has $u' \geq u$ sets S_i and at most $u' + 1$ vertices, or at most $1 + 1/u$ vertices per part in A. The number of vertices in G'_A is thus at most $k(1 + 1/u)$. The total number of vertices in G is thus at most $k\ell - k + u + k + k/u + 1 \leq k\ell + k + 2 < (k+1)(\ell+1)$. □

3 Class Two Matrices

We now consider normal matrices of class two, i.e., assume that $\mathcal{C}$ is either $\{*\}$ or a subset of $\{0, 1\}$. In these cases we can only prove the existence of large obstructions. This should not be surprizing, as we have already seen an example with $\mathcal{A} = \mathcal{B} = \mathcal{C} = \{*\}$, where there are minimal obstructions with strictly more than $(k+1)(\ell+1)$ vertices. More general examples are described in the next section.

All our results in these cases are applications of the following general theorem.

Let $\mathcal{S}$ and $\mathcal{D}$ be two classes of graphs, each closed under taking induced subgraphs, and such that the largest graph that is both in $\mathcal{S}$ and in $\mathcal{D}$ has at most r vertices. An *$\mathcal{S}, \mathcal{D}$-partition* of a graph G is a partition of the vertices of G into two sets A and B, such that A induces a subgraph G_A of G that is in $\mathcal{S}$, and B induces a subgraph G_B of G that is in $\mathcal{D}$. In [10] we studied these partitions (called *sparse-dense* partitions), and proved that if the classes $\mathcal{S}, \mathcal{D}$ have polynomial time recognition algorithms, then so does the class of $\mathcal{S}, \mathcal{D}$- partitionable graphs. Here we prove that, moreover, if both classes $\mathcal{S}$ and $\mathcal{D}$ have bounded minimal obstruction size, then so does the class of $\mathcal{S}, \mathcal{D}$-partitionable graphs.

To be specific, a *minimal $\mathcal{S}$-obstruction* is a graph G that is not in $\mathcal{S}$, but every proper induced subgraph of G is in $\mathcal{S}$. *Minimal $\mathcal{D}$-obstructions* are defined analogously. Finally, a minimal $\mathcal{S}, \mathcal{D}$- obstruction is a graph G that is not $\mathcal{S}, \mathcal{D}$- partitionable, but every proper induced subgraph of G is $\mathcal{S}, \mathcal{D}$-partitionable.

More generally, let $\mathcal{P}$ be any property of $\mathcal{S}, \mathcal{D}$-partitions of graphs that is preserved under taking induced subgraphs and induced partitions. (In other words, if G with partition A, B has the property, and if G' is a subgraph of G induced by the set V' of vertices, then G' with the partition $V' \cap A, V' \cap B$ also has the property.) A *$\mathcal{P}$-partition* of a graph G is an $\mathcal{S}, \mathcal{D}$-partition of G which has property $\mathcal{P}$. A *minimal $\mathcal{P}$-obstruction* is a graph G which does not admit a *$\mathcal{P}$-partition*, but such that every proper induced subgraph does admit a *$\mathcal{P}$-partition*. A *minimal $\mathcal{P}$-obstructed partition* is a graph G and a partition A, B which is not a *$\mathcal{P}$-partition* (in other words, $A \notin \mathcal{S}$, or $B \notin \mathcal{D}$, or A, B does not have the property P), but such that every proper induced subgraph of G with the induced partition is a $\mathcal{P}$-partition of G.

Theorem 3.1 *Suppose for each a minimal $\mathcal{P}$-obstructed partition G, A, B the graph G has at most p vertices. Then each minimal $\mathcal{P}$-obstruction has at most $2p^{2r+1}$ vertices.*

Proof. Let G be a minimal $\mathcal{P}$-obstruction, and let v_0 be any vertex of G. The graph $G - v_0$ has an $\mathcal{P}$-partition into subgraphs $G_A \in \mathcal{S}$ and $G_B \in \mathcal{D}$. Imagine trying to certify that G is not $\mathcal{P}$-partitionable. In any $\mathcal{P}$-partition of G into A', B', we would have $|A \cap B'| \leq r$ and $|A' \cap B| \leq r$, by the definition of r. Thus G is certified as non-partitionable by any set of $2r + 1$ distinct vertices that are guaranteed to be in $(A \cap B') \cup (A' \cap B)$, for any $\mathcal{P}$-partition A', B' of G. We now consider how much of the graph G is needed to obtain such a certificate of $2r+1$ vertices. Assume that A', B' is any $\mathcal{S}, \mathcal{D}$-partition of G, and suppose that $v_0 \in A'$.

Consider the partition of G into sets $A \cup v_0$ and B. The sets $A \cup v_0, B$ do not give a $\mathcal{P}$-partition, since G is not $\mathcal{P}$-partitionable. Thus G contains an induced subgraph G' with at most p vertices that is not $\mathcal{P}$-partitioned. Then some vertex v_1 in G' must be in $(A \cap B') \cup (A' \cap B)$, since A', B' are assumed to give a $\mathcal{P}$-partition Now we have a new partition of G into sets $A \cup v_0 - v_1$ and $B \cup v_1$, or into sets $A \cup v_0 \cup v_1$ and $B - v_1$. Thus again we do not have a valid $\mathcal{P}$-partition, which gives again an induced subgraph G'' with at most p vertices that is not $\mathcal{P}$-partioned. Again, one of these vertices must be moved from the one set to the other, giving us a choice of v_2. Continuing this way, we obtain a sequence of vertices $v_1, \ldots, v_{2r+1}$, all distinct, and all in $(A \cap B') \cup (A' \cap B)$.

The vertex v_0 has two choices, going to A or going to B. At each iteration we select at most $p-1$ possible choices for v_i to be moved from A to B or from B to A, since the obstruction of size p must contain either v_0 or some vertex moved earlier, and the number of iterations is $2r+1$, thus identifying an induced subgraph H of G that is not partitionable with at most $1 + 2((p-1) + (p-1)^2 + \ldots + (p-1)^{2r+1}) < 2p^{2r+1}$ vertices. □

We now apply the theorem to our M-partition problems. For simplicity, when we state the bounds in this section, we shall assume that $k \leq \ell$, by complementing M if necessary.

Corollary 3.2 *Assume $\mathcal{C} = \{*\}$, and $\mathcal{A}, \mathcal{B}$ are any normal sets.*

Then each perfect graph which is a minimal obstruction to M-partitionability has at most $2(\ell+1)^{2k\ell+1}$ vertices.

Proof. We take as $\mathcal{S}$ the class of perfect graphs that admit an A-partition (A is the diagonal k by k block of M), and as $\mathcal{D}$ the class of perfect graphs that admit a B-partition. Since A-partitionable graphs are k-colourable, and B-partitionable graphs have ℓ-colourable complements, the size of graphs G in $\mathcal{S} \cap \mathcal{D}$ is bounded by $r = k\ell$. Indeed, such a graph G is k-colourable and has all independent sets (and hence all colour-classes) of size at most ℓ. It remains to bound the size of perfect minimal $\mathcal{S}$- and $\mathcal{D}$-obstructions: When $\mathcal{A} = \{*\}$, there is only one minimal obstruction – the clique with $s = k+1$ vertices. When $\mathcal{A} \subseteq \{0,1\}$, we apply Theorem 2.1 to deduce that each obstruction has at most $s \leq k+1$ vertices. Similarly, $\mathcal{B} = \{*\}$ yields $t = \ell + 1$, and $\mathcal{B} \subseteq \{0,1,\}$ also yields $t \leq \ell + 1$. Since $r = k\ell$, we derive the claimed bound from Theorem 3.1. □

We note that we can again weaken the assumption of perfectness to the assumption that the absence of clique of size $k+1$ implies k-colourability, needed only when $\mathcal{A} = \{*\}$, and the assumption that the absence of an independent set of size $\ell+1$ implies ℓ-colourability of the complement, needed only when $\mathcal{B} = \{*\}$.

Lemma 3.3 *Assume $\mathcal{C} = \{0,1\}$, and $\mathcal{A}, \mathcal{B}$ are any normal sets.*

Let G be a graph whose vertices have beeen partitioned into a subset to go to G_A and a subset to go to G_B. If $\mathcal{A} = \{*\}$ or $\mathcal{B} = \{*\}$, also assume that G is a perfect graph.

If G has no M-partition conforming to this assignment, then G has a minimal obstruction of size at most $2\ell+1$.

Proof. Partition all vertices of G in A into subsets A_i of vertices that have the same set of neighbors in G if $\mathcal{A} \subseteq \{0,1\}$, or that have the same set of neighbors in B if $\mathcal{A} = \{*\}$. Similarly, partition all vertices of G in B into subsets B_i of vertices that have the same set of neighbors in G if $\mathcal{B} \subseteq \{0,1\}$, or that have the same neighbors in A if $\mathcal{B} = \{*\}$.

Partition each subset A_i into subsets A_{ij} by using just a single $A_{ij} = A_i$ if $\mathcal{A} = \{0,1\}$ and A_i is an independent set, by using $|A_i|$ single vertex subsets A_{ij} if $\mathcal{A} = \{0,1\}$ and A_i is a clique, and by using r_i subsets A_{ij} that each induce an independent set in G if $\mathcal{A} = *$. Here r_i is both the the largest clique size and the chromatic number for the subgraph of G induced by A_i. Similarly, partition each subset B_i into subsets B_{ij} by using just a single $B_{ij} = B_i$ if $\mathcal{B} = \{0,1\}$ and B_i is a clique, by using $|B_i|$ single vertex subsets B_{ij} if $\mathcal{B} = \{0,1\}$ and B_i is an independent set, and by using s_i subsets B_{ij} that each induce a clique in G if $\mathcal{B} = \{*\}$. Here s_i is both the the size of the largest independent set in, and the minimum number of cliques covering, the subgraph of G induced by B_i.

Define G' as the subgraph of G induced by a subset V' of the vertices of G that contains exactly one vertex from each A_{ij} and from each B_{ij}, such that if $\mathcal{A} = \{*\}$ then the vertices in each $V' \cap A_i$ form a clique in G, and if $\mathcal{B} = \{*\}$ then the vertices in each $V' \cap B_i$ form a clique in G. Every solution for G' can be extended to a solution for G by assigning all vertices in a set A_{ij} or B_{ij} to the same part as their chosen representative in V'. So if G is an obstruction, then so is G'. Furthermore, all vertices in V' must go to different parts.

If G' has at least $k+1$ vertices in A, then choose $k+1$ vertices of G' in A, and for every pair x, y, of two such vertices, if x, y do not share an edge, choose a vertex z in G' that is a neighbor of exactly one of x, y, with z in B

if $\mathcal{A} = \{*\}$. As each chosen z partitions the $k+1$ chosen vertices in A into two sets, the neighbors and the non-neighbors, only k vertices z are needed, giving an obstruction with at most $2k+1$ vertices. Similarly, if G' has at least $\ell+1$ vertices in B, then choose $\ell+1$ vertices of G' in B, and for every pair x, y, of such vertices, if x, y do share an edge, choose a vertex z in G' that is a neighbor of exactly one of x, y, with z in A if $\mathcal{B} = \{*\}$. As each chosen z partitions the $\ell+1$ chosen vertices in B into two sets, the neighbors and the non-neighbors, only ℓ vertices z are needed, giving an obstruction with at most $2\ell+1$ vertices. In the remaing case G' has at most k vertices in A and ℓ vertices in B, giving an obstruction with $k+\ell$ vertices. □

Corollary 3.4 *Assume $\mathcal{C} = \{0,1\}$, and $\mathcal{A}, \mathcal{B}$ are any normal sets.*

Each perfect graph which is a minimal obstruction to M-partitionability has at most $2(2\ell+1)^{2k\ell+1}$ vertices.

Proof. This follows from Theorem 3.1 by Lemma 3.3 as in the proof of Corollary 3.2, since by Lemma 3.3, the minimal obstructions for a given M-partition have size at most $2\ell+1$, and as in Corollary 3.2, a graph G that can be assigned both to A and to B has size at most $k\ell$. □

4 Conclusions

Our main conclusion is the following corollary of the preceeding results:

Theorem 4.1 *If M is a normal matrix, then there are only finitely many perfect minimal obstructions to M-partitionability.*

We now explain why we think it will be hard to completely classify the complexity of all M-partition problems for perfect graphs:

Theorem 4.2 *Every constraint satisfaction problem in CSP is polynomial time equivalent to an M-partition problem for perfect graphs.*

It follows from the theorem that even just proving that every M-partition problem for perfect graphs is polynomial or NP-complete would solve the longstanding open problem of dichotomy for constraint satisfaction problems.

Proof. As mentioned in the introduction, every constraint satisfaction problem is polynomial time equivalent to a homomorphism problem to a digraph H. Thus it will suffice to prove that for every digraph H, the problem of existence of homomorphisms to H, also known as the *H-colouring problem* [14], is polynomial time equivalent to some M-partition problem. In the H-colouring problem we are asked whether or not an input digraph G admits a homomorphism, i.e., an arc-preserving mapping, to H. (All digraphs in this proof will be assumed to have no loops; otherwise the homomorphism problem is trivial.)

Thus let H be a fixed digraph. We shall first describe the matrix M corresponding to H:

Let $k = 4|V(H)|$ and $\ell = 4$. Let A have four parts $p_1(v)$, $p_2(v)$, $p_3(v)$, $p_4(v)$, for each $v \in V(H)$, and let B have four parts q_1, q_2, q_3, q_4. The $*$ entries in A are $p_1(v)p_2(v)$, $p_2(v)p_3(v)$, $p_3(v)p_4(v)$, and $p_1(v)p_4(w)$ for all vertices v, w of H with $vw \in E(H)$; all other entries of A are 0. The 1 entries in C are $p_1(v)q_1$, $p_2(v)q_2$, $p_3(v)q_3$, $p_4(v)q_4$; all other entries in C are 0.

The H-colouring problem has a polynomial time reduction to the M-partition problem for perfect graphs.

Given an digraph G, an instance of the H-colouring problem, we define a perfect graph G', an instance of the M-partition problem as follows: G' has two vertices r_i, s_i for each part q_i, and four vertices $p'_1(v)$, $p'_2(v)$, $p'_3(v)$, $p'_4(v)$ for each vertex v in $V(G) \cup V(H)$. The edges of G' are $p'_1(v)p'_2(v)$, $p'_2(v)p'_3(v)$, $p'_3(v)p'_4(v)$, and $p'_1(v)p'_4(w)$ whenever $vw \in E(G) \cup E(H)$, plus the triangles $p'_i(v)r_is_i$.

It can be checked that G admits a homomorphism to H if and only if G' admits an M-partition. We will omit the details.

The M-partition problem for perfect graphs also has a polynomial time reduction to the H-colouring problem for digraphs. Indeed, suppose we are given a perfect graph G', as an instance of the M-partition problem. We may test in polynomial time whether an induced subgraph G'' of G' admits an A-partition, since this is equivalent to G'' being bipartite, and also test whether G'' admits a B-partition, since this is equivalent to G'' consisting of at most four independent cliques if $\mathcal{B} = \{0\}$, and is equivalent to G'' not having five independent vertices (which is equivalent to the complement of G'' being four-colourable by perfectness). We thus have polynomial tests for memberships in $\mathcal{S}$ consisting of all A-partitionable graphs, and in $\mathcal{B}$ consisting of all B-partitionable graphs. Therefore, the algorithm of Feder, Hell, Klein, and Motwani [10] produces all the polynomially many partitions of G' into an A-partitionable subgraph G'_A and a B-partitionable subgraph

G'_B. Consider each such partition of G'. The bipartite subgraph R of G' consisting of the edges joining G'_A and G'_B must have edges forming independent complete bipartite subgraphs plus isolated vertices, by the choice of C. If there are at least five such independent complete bipartite subgraphs then there is no solution. If there are four such independent complete bipartite subgraphs and at least one additional isolated vertex then there is no solution. If R consists of just four independent complete bipartite subgraphs, then we try each labelling of these complete bipartite subgraphs as R_1, R_2, R_3, R_4, and test whether we may place the vertices of R_i from parts in B to q_i, and satisfy the conditions required by B. If this succeeds, we must place the vertices of R_i from parts in A to some $p_i(v)$. We may identify any two vertices of each R_i in A that have a common neighbor in R_{i+1} or a common neighbor in R_{i-1}. Once this is done, we must place each path $u_1u_2u_3u_4$ in some $p_1(v)$ $p_2(v)$ $p_3(v)p_4(v)$, and choose $f(u_1) = p_1(v)$ and $f(u'_4) = p_4(w)$ for each edge $u_1u'_4$ of G' so that vw is an arc of H. Thus the paths $u_1u_2u_3u_4$ correspond to vertices v of a digraph G that has arcs vw that must map to arcs of H, that is, G is an instance of the digraph H-colouring problem, and G admits a homomorphism to H if and only if G' admits an M-partition.

Similar arguments comprise the polynomial time reduction in the remaining cases where R consists of three or fewer independent complete bipartite subgraphs plus possibly some isolated vertices. □

(The details omitted in the above proof are available from the authors.) It can be seen that this proof in fact shows that the M-partition problem for a general graph G' reduces to the H-colouring problem for digraphs. It can also be seen that the matrix M can be chosen to have $\mathcal{A} = \{0, *\}$, and $\mathcal{B} = \{0\}$, or alternately $\mathcal{B} = \{*\}$ and $\mathcal{C} = \{0, 1\}$.

Next, we describe an example when the linear bound $O(k\ell)$ on the size of perfect minimal obstructions does not apply to normal matrices of class one. The matrices M we use have $k = 1$, $\mathcal{B} = \{0\}$, and $\mathcal{C} = \{*\}$.

We shall show that (for infinitely many values ℓ) there are perfect graphs (in fact, trees) which are minimal obstructions and have at least $(\ell/3)^2$ vertices.

Let p be a positive integer. For each $t \geq 2^p$, we shall construct a tree obstruction R with $v = 3\cdot 2^p - 3 + p\cdot 2^p + t\cdot 2^p + t$ vertices and no M-partition with $\ell < r = 2^{p+1} - 2 + t$, but such that every proper induced subgraph of R has such an M-partition. Thus R is a minimal obstruction for $\ell = r - 1$. If $t = 2^p$ and $\ell = r - 1$ then $\ell \leq 3t$ and $v \geq t^2 \geq (\ell/3)^2$.

Let R' be an auxiliary tree in which the root (at depth 0) has two children, and in general every vertex at depth $2i < 2p$ has two children, while each vertex at depth $2i-1 < 2p-1$ has just one child. Each vertex at level $2p-1$ is a leaf. It is easy to see that the tree R' has $3 \cdot 2^p - 3$ vertices, has an M-partition with $\ell = 2^p - 1$, in which all leaves are in cliques and the root is in the independent set, and an M-partition with $\ell = 2^{p+1} - 2$, in which all leaves are in the independent set and the root is in a clique. It can also be seen that R' has no M-partition with $\ell < 2^p - 1$ or $\ell > 2^{p+1} - 2$. Furthermore, for every leaf x of R', there is an M-partition with $\ell = 2^{p+1} - 2 - p$ that puts x in the independent set and all other leaves in cliques, and puts the root in a clique. Let X denote the set consisting of the root of R' and all the leaves of R'. Note that for any $x \in X$ there is a partition that puts x in the independent set and all other $y \in X$ is cliques. finally, we note that if we remove any vertex from R', then there is an M-partition that puts all remaining vertices of X in cliques.

We now define the tree R by letting $t \geq 2^p$ and attaching t new vertices to the root, and $t + p$ vertices at each of the leafs of R'. The tree R has $v = 3 \cdot 2^p - 3 + p \cdot 2^p + t \cdot 2^p + t$ vertices. The partitions of R' that put $x \in X$ in the independent set and all $y \in X$ in cliques, add t vertices in cliques if x is the root, and add $t + p$ vertices in cliques if x is a leaf; thus these M-partitions have $\ell = r = 2^{p+1} - 2 + t$ cliques. Any other partition of R has at least $2^p - 1$ cliques in R', and at least $2t + p$ cliques at the new leaves; thus it has $\ell \geq 2^p - 1 + 2t + p > r$ cliques. If we remove a vertex of R', then there is a partition of R where the remaining vertices out of X are in cliques, with $\ell \leq 2^{p+1} - 2 < r$.

Thus there is no M-partition of R with $\ell < r$, yet R minus any vertex has an M-partition with $\ell < r$. Therefore R is a minimal obstruction for $\ell = r - 1$. Finally, we note that when $t = 2^p$ the tree R has $v \geq (\ell/3)^2$ vertices.

We know from Corollary 3.2 that this M-partition problem still has only finitely many perfect minimal obstructions. For chordal graphs we have established the correct order of the largest minimal obstruction to be $O(\ell^2)$ [11]. (Other bounds are also lowered for chordal graphs in [11].)

Our last remark concerns algorithms for the M-partition problems discussed here. From the proof of the theorems in the preceeding sections, we can extract polynomial time algorithms for the corresponding M-partition problems on perfect graphs. (Remember that M is fixed and not part of the instance.) All our algorithms are *certifying* algorithms, in that they identify either a partition or an obstruction, in the stated time. In fact, our algorithms actually proceed by detecting the existence of obstructions, along the lines of the proofs. The most computationally expensive part of the algorithms consists of finding cliques of size $k+1$ in the cases with $\mathcal{A} = \{*\}$, and independent sets of size $\ell+1$ in the cases with $\mathcal{B} = \{*\}$. With the use of the ellipsoid method, all the results of Section 2 on normal matrices of class one have algorithms of complexity $O(f(k,\ell)n^c)$, where the constant c does not depend on k and ℓ. The same holds in Section 3 for Lemma 3.3. However, finding an $\mathcal{S},\mathcal{D}$-partition in the two corollaries in Section 3 takes time $O(n)^{O(k\ell)}$.

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