On Thinness of Spanning Trees in Planar and Series-Parallel Graphs

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1 Results and Proofs

Given a graph $G = (V, E)$ and a set $S \subset V$, we denote by $E(S, \overline{S})$ the edges across the cut from S to $\overline{S} = V \setminus S$. If $F \subseteq E$, we denote by $F(S, \overline{S})$ the set $F \cap E(S, \overline{S})$. For $\alpha \geq 0$, we say that F is *α-thin*, or has *thinness* α, if for all $S \subseteq \alpha$, $|F(S, \overline{S})| \leq \alpha |E(S, \overline{S})|$. We are particularly interested in thinness when $F = T$ is a spanning tree.

Thinness was studied [1] in the special case where $F = T$ is a spanning tree and G is a k-edge-connected multigraph (parallel edges allowed) of bounded genus. Here we focus on k-edge connected multigraphs that are planar or series-parallel.

If G is an embedded planar multigraph, we say that edges $e_1, \ldots, e_r$ are strictly parallel if they all have the same two endpoints u, v, and each pair e_i, e_{i+1} forms a face that is a digon. Note that parallel edges may not form a single strictly parallel set.

The following follows from Euler's formula.

Lemma 1.1 *Every multigraph G without strictly parallel edges (but possibly with parallel edges that are not strictly parallel) has a vertex v of degree at most 5 none of whose incident edges are parallel.*

Theorem 1.2 *If G is a k-edge-connected planar multigraph, then G has a spanning tree T that can be found in polynomial time that has thinness $\alpha = 1/\lceil k/10 \rceil$.*

Proof. Let G' be the multigraph obtained from G by replacing strictly parallel edges with a single edge. Consided the vertex v of degree at most 5 without incident parallel edges from Lemma 1.1. The vertex v has degree at least k by k-edge-connectivity in G. Since these at least k edges incident to v in G correspond to at most 5 edges in G', we conclude that some edge e' incident to v in G' must correspond to $m \geq \lceil k/5 \rceil$ strictly parallel edges incident to v in G. Select the middle edge e out of these m, in position $\lceil m/2 \rceil \geq \lceil k/10 \rceil$, and include e in T. Then contract all the edges strictly parallel to e in G, corresponding to e' in G', and proceed by induction until the whole graph G is retracted to a single vertex, in which case the iteratively selected edges will form the tree T.

A cut $E(S, S')$ in the graph G is given by a union of vertex-disjoint cycles in the dual graph G^* of G. We may prove thinness α for T across all cuts by assuming the cut $E(S, S')$ is a single such cycle C in G^*, as combining various cycles preserves thinness α.

Label the edges that were contracted in the $n-1 = |V|-1$ stages that defined T by $1, 2, \ldots, n-1$ in order. The cycle C in G^* then traverses a series of edges given by numbers $a_1, a_2, \ldots, a_r$, where we do not repeat numbers in the same stage, that is $a_i \neq a_{i+1}$. For some a_i, visiting edge e contracted in stage a_i also visits the special edge e' in T out of the edges corresponding to e. If $a_{i-1} > a_i$ (resp. $a_{i+1} > a_i$) and e' is visited, then between a_{i-1} (resp a_{i+1}) and e of a_i we will visit $\lceil m/2 \rceil - 1 \geq \lceil k/10 \rceil - 1$ strictly parallel edges corresponding to e in a_i before (resp. after) e' in T. On the other hand, if $a_{i-1} < a_i$ (resp. $a_{i+1} < a_i$)) then the various decreasing (resp. increasing) j starting at $i-1$ until we reach an edge of T at j' may continue to decrease a_j, in which case when we reach $a_{j'}$ we get again $\lceil m/2 \rceil - 1 \geq \lceil k/10 \rceil - 1$ strictly parallel edges for $a_{j'}$ as before for a_i. Finally, if in this process of decreasing (resp. increasing) j, the value a_j decreases and then increases before reaching j, then we have $a_{j-1} > a_j < a_{j+1}$ and thus again $m \geq \lceil k/5 \geq \lceil k/10 \rceil - 1$ strictly parallel edges for a_j between edges of T along C. Thus only one out of at least $\lceil k/10 \rceil$ consecutive edges of C is in T, proving thinness $\alpha = 1/\lceil k/10 \rceil$. □

Corollary 1.3 *If G is a k-edge-connected series-parallel multigraph, then G has a spanning tree T that can be found in polynomial time that has thinness $\alpha = 1/\lceil k/4 \rceil$.*

Proof. This follows as in the proof of the planar case, except that the degree 5 for planar multigraphs in Lemma 1.1 is improved to degree 2 for series-parallel multigraphs. Therefore $m \geq \lceil k/5 \rceil$ is replaced by $m \geq \lceil k/2 \rceil$, and $\lceil m/2 \rceil \geq \lceil k/10 \rceil$ is consequently replaced by $\lceil m/2 \rceil \geq \lceil k/4 \rceil$. □

Similar techniques give thinness $f(g)/k$ for graphs of bounded genus g and some positive integer function f, see also [1].

A graph G is said to be of *treewidth* r if we are given a tree R such that each vertex v of G corresponds to a subtree R_v of R in such a way that any two adjacent u, v in G have R_u, R_v sharing a vertex in R and the number of R_v that meet at a vertex of R is at most $r+1$. It remains open whether thinness $f(r)/k$ can be achieved for graphs of treewith r and some positive integer function f.

References

[1] Shayan Oveis Gharan and Amin Saberi, The asymmetric traveling salesman problem on graphs with bounded genus, SODA 2011.