List homomorphisms and retractions to reflexive digraphs

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Abstract

We study the list homomorphism and retraction problems for the class of reflexive digraphs (digraphs in which each vertex has a loop). These problems have been intensively studied in the case of undirected graphs, but the situation seems more complex for digraphs. We also focus on an intermediate 'subretraction' problem. It turns out that the complexity of the subretraction problem can be classified at least for large classes of reflexive digraphs; by contrast, the complexity of the retraction problem for reflexive digraphs seems difficult to classify. For general list homomorphism problems, we conjecture that the problem is NP-complete unless H is an 'adjusted' interval digraph, in which case it is polynomial time solvable. We prove several cases of this conjecture. The class of adjusted interval digraphs appears interesting in its own right.

1 Introduction

A digraph H is *reflexive* if the adjacency relation $E(H)$ is reflexive, i.e., if each vertex has a loop; it is *symmetric* if the relation $E(H)$ is symmetric, i.e., if $uv \in E(H)$ implies $vu \in E(H)$; and it is *antisymmetric* if the relation $E(H)$ is antisymmetric, i.e., if $uv \in E(H)$ implies $vu \notin E(H)$.

Each digraph H is associated with two related undirected graphs. We denote by $U(H)$ the *underlying graph* of H, which has an edge uv whenever $u \neq v$ and $uv \in E(H)$ or $vu \in E(H)$, and by $S(H)$ the *symmetric graph* of H, which has an edge uv whenever $u \neq v$ and $uv \in E(H)$ and $vu \in E(H)$. We shall say that u is a *neighbour* of v in H, and that u, v are *adjacent* in H, if uv is an edge of $U(H)$. Note that the loops of H, if any, are removed from both $U(H)$ and $S(H)$.

A graph is *chordal* if it does not contain an induced cycle of length at least four. An *interval graph* is a graph H which admits an *interval representation*, i.e., a family of intervals $I_v, v \in V(G)$, such that $uv \in E(H)$ if and only if I_u and I_v intersect.

While a digraph H is called *chordal* whenever its underlying graph $U(H)$ is chordal, there is a specialized notion of interval digraph [24, 26]. An *interval digraph* is a digraph H which admits an *interval pair representation*, which is a family of pairs of intervals $I_v, J_v, v \in V(G)$, such that $uv \in E(H)$ if and only if I_u intersects J_v. For most other undirected concepts, such as a tree or a cycle, we usually say that a digraph H has the property (is a tree or a cycle etc.) if the underlying graph $U(H)$ has the property.

If $uv \in E(H)$, then uv is an *edge of* H. If $uv \in E(H)$ and $vu \in E(H)$, we say that uv is *double edge* (or *symmetric edge*). If $uv \in E(H)$ but $vu \notin E(H)$, we say that uv is a *forward edge*. We additionally use the term a *backward edge* for an edge uv of $U(H)$ such that $vu \in E(H)$ but $uv \notin E(H)$. Note that while a forward edge is an actual edge of H, a backward edge is an edge of $U(H)$ but not of H. A *single edge* is a forward or backward edge.

A *homomorphism* f of a digraph G to a digraph H is a mapping $f : V(G) \to V(H)$ in which $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$ [17]. If $L(v), v \in V(G)$, are *lists* (subsets of $V(H)$), then a *list homomorphism* of G to H (with respect to the lists L) is a homomorphism satisfying $f(v) \in L(v)$ for all $v \in V(G)$. If H is a subgraph of G, a *retraction* of G to H is a list homomorphism of G to H with respect to lists L in which $L(u) = \{u\}$ for all $u \in V(H)$, and $L(u) = V(H)$ for all $u \in V(G) \setminus V(H)$.

Let H be a fixed digraph H. The basic *homomorphism problem* $HOM(H)$ asks whether or not an input digraph G admits a homomorphism $f : G \to H$. (Note that this problem is trivial if H is reflexive.) The *list homomorphism* *problem* $L - HOM(H)$ asks whether or not an input digraph G equipped with lists L admits a list homomorphism $f : G \to H$ with respect to L. Note that the basic homomorphism problem is a restriction of the list homomorphism problem to input graphs G with lists L in which each $L(v), v \in V(G)$ is the entire set $V(H)$. The *retraction problem* $RET(H)$ asks whether or not an input digraph G containing H as a subgraph admits a retraction to H. It is easy to see, cf. [3], that this problem is equivalent to the restriction of $L - HOM(H)$ to input graphs G with lists L in which each list $L(v), v \in V(G)$, is either a single vertex of H or the entire set $V(H)$. The *subretraction problem* $SubRET(H)$ defined as the slightly weaker restriction of $L - HOM(H)$ to input graphs G with lists L in which each list $L(v), v \in V(G)$, is either a single vertex of H or the same set S for some $S \subseteq V(H)$. A final restriction we shall sometimes consider is the *connected lists homomorphism problem* $CL - HOM(H)$, in which the input graphs G with lists L are only only restricted to have each list $L(v), v \in V(G)$, induce a connected subgraph of $U(H)$.

We shall always assume that the graph $U(H)$ is connected. It then follows from these definitions that $RET(H)$ is a restriction of $SubRET(H)$, which in turn can be viewed as a restriction of $CL - HOM(H)$ (as we may ignore the situations when the no-trivial lists induce a disconnected graph). Of course, all the problems we discussed are restrictions of the list homomorphism problem $L - HOM(H)$, and all contain the basic homomorphism problem $HOM(H)$.

Except for $SubRET(H)$, these problems have been studied in the case of (undirected) graphs [17]. For instance, the first two authors have proved in [3] that for reflexive graphs H, the problem $L - HOM(H)$ is polynomial time solvable when H is an interval graph, and is NP-complete otherwise. It is also proved in [3] that for reflexive graphs proving dichotomy of $RET(H)$ (i.e., showing that each $RET(H)$ is NP-complete or polynomial time solvable) is equivalent to proving such a dichotomy for all constraint satisfaction problems, which appears to be difficult [8]. However, dichotomy of the problems $CL - HOM(H)$ for reflexive graphs H has been proved, and the complexity fully classified [3]: if H is chordal, the problem is polynomial time solvable, otherwise it is NP-complete. The same classification also applies to the more restrictive problem $SubRET(H)$. Indeed, when H is not chordal, the problem $SubRET(H)$ remains NP-complete, as it contains the problem $RET(H')$, where H' is a reflexive chordless cycle, proved NP-complete in [3]. Thus we obtain the following dichotomy classification of $SubRET(H)$ for reflexive graphs.

Theorem 1.1 *For reflexive graphs H, the problem $SubRET(H)$ is polynomial time solvable if H is chordal, and it is NP-complete otherwise.*

In this paper we investigate the analogues of these results for reflexive digraphs, focusing primarily on $L-HOM(H)$ and $SubRET(H)$. The above remarks about graphs directly imply the following facts about digraphs:

- *If H is a reflexive digraph such that $S(H)$ is not an interval graph, then the problem $L-HOM(H)$ is NP-complete.*
- *If H is a reflexive digraph such that $S(H)$ is not chordal, then the problem $SubRET(H)$ is NP-complete.*

In the next two sections we shall extend the above facts to $U(H)$ in place of $S(H)$, i.e., we shall prove that $L-HOM(H)$ is NP-complete unless $U(H)$ is an interval graph, and $SubRET(H)$ is NP-complete unless $U(H)$ is a chordal graph.

We have also studied the complexity of problems $L-HOM(H)$ when H is an irreflexive undirected graph [4], or an arbitrary undirected graph with loops allowed [5]. In each case, we were able to obtain a classification - the problems tend to be tractable for well structured and natural classes of graphs, and NP-complete otherwise. The complexity of $L-HOM(H)$ for any digraph (or more general relational system) has been classified in [1]. The classification is complicated, but it does yield an algorithm to decide for any fixed digraph H whether $L-HOM(H)$ is polynomial time solvable or NP-complete. We will propose a simpler graph theoretic characterization of the tractable problems, (similar to that for undirected graphs [3, 4, 5]) and verify it for large classes of digraphs H - including trees and certain orientations of complete graphs. Note that trees and complete graphs are the building blocks of all interval graphs [9], so these are important special cases to consider. If our conjecture is true, it represents a significant simplification of the classification in the special case of reflexive digraphs. The tractable cases of $L-HOM(H)$ would again correspond to nicely structured digraphs H. Moreover, they would also correspond to just one simple ordering property - min-orderability - of digraphs. We also have a similar conjecture for the special case of irreflexive digraphs: there we believe, in addition to min-orderability, one needs also to check for the existence of a majority function [14]. (The statement in [14] erroneously only lists majority function; both majority and min-orderability are needed [7].)

For $SubRET(H)$, we present two classes of chordal digraphs for which the problem is polynomial time solvable - namely trees and chordal partial orders. We also have reflexive chordal digraphs with NP-complete subretraction problems, including the directed three-cycle. Still, the complexity of the subretraction problem can be classified at least for large classes of reflexive digraphs. By contrast, the complexity of the retraction problem for reflexive digraphs seems again difficult to classify [3, 8].

2 Chordal digraphs

In this section we prove a basic result about cycles that has implications for all reflexive digraph list homomorphism problems. (Recall that according to our convention a cycle is a digraph H such that $U(H)$ is an undirected cycle.) We begin by observing that the results of [6] imply that for each reflexive digraph H with up to three vertices, the problem $L-HOM(H)$ is polynomial time solvable, with the sole exception of the directed three-cycle C_3, for which the problem $RET(C_3)$ is NP-complete. These facts classify the complexity of the list homomorphism, retraction, and subretraction problems for all cycles H of length three or less. For cycles H of length greater than three, we shall show that $RET(H)$ is always NP-complete. (This was independently proved by Benoit Larose (personal communication), using a result of [2].)

For our proof we will employ the *indicator construction* from [16], as explained in [17]. For a fixed *indicator* I, i, j (that is a digraph I with two specified vertices i, j), the indicator construction transforms a digraph H into the digraph H^*, with the same vertex set as H, and with adjacency defined by the following rule: xy is an edge of H^* just if there exists a homomorphism of I to H that maps i to x and j to y. It is easy to see that the following extension of Lemma 5.5 of [17] holds. (The proof is identical, with the trivial addition of singleton lists wherever they were present in the given instance.)

Lemma 2.1 *If the problem $RET(H^*)$ is NP-complete, then so is the problem $RET(H)$.*

For future reference we remark that the indicator construction is also useful for list homomorphism problems. Let I, i, j be an indicator in which each vertex v has a list $L(v) \subseteq V(H)$. The digraph H^* is now defined to have an edge xy just if there is a list homomorphism of I to H mapping i to x and j to y. The following observation is proved exactly as Lemma 2.1; note that we assume that the lists of the vertices i and j are the entire $V(H^*)$:

this ensures that the proof in [17] properly applies to reduce $L-HOM(H^*)$ to $L-HOM(H)$.

Lemma 2.2 *If the problem $L-HOM(H^*)$ is NP-complete, and if $L(i) = L(j) = V(H^*)$, then the problem $L-HOM(H)$ is also NP-complete.*

The second tool for our proof involves associating with each digraph H a bipartite graph $B(H)$, in which each vertex $v \in V(H)$ yields two vertices v', v'' in $B(H)$ and each edge $vw \in E(H)$ yields an edge $v'w''$ in $B(H)$. Note that if H is reflexive, we have $v'v'' \in E(B(H))$ for each $v \in V(H)$.

Lemma 2.3 *bigs If $RET(B(H))$ is NP-complete then $RET(H)$ is also NP-complete.*

Proof. Consider a graph G with lists (that are singletons or the entire set $V(B(H))$). If G is not bipartite, there is no list homomorphism. Else we may assume we have vertices that have lists from the set of all $v', v \in V(H)$ (call these white vertices), and vertices that have lists from the set of all $v'', v \in V(H)$ (call these black vertices). We can transform G to a digraph G' with $V(G') = V(G)$ obtained by orienting each edge of G from the white to the black vertex. The lists in G' are obtained from the lists of G by dropping the primes and double primes. It is easy to see that G admits a list homomorphism to $B(H)$ if and only if G' admits a list homomorphism to H. $\square$

We again note that the same proof reduces $L-HOM(B(H))$ to $L-HOM(H)$ (cf. [7, 11, 10]).

Theorem 2.4 *If H is a reflexive digraph such that $U(H)$ is a cycle with at least four vertices, then the retraction problem $RET(H)$ is NP-complete.*

Proof. Let the consecutive vertices of H be $a_1a_2\cdots a_pa_1$, $p \geq 4$. Thus for each t, the vertex a_t has a loop, there is an edge $a_ta_{t+1} \in E(U(H))$ which may be forward, backward, or double; and H has no other edges. (Addition in cycle subscripts is always taken modulo the length p.)

Firstly, we observe that if all edges a_ta_{t+1} are double, then $RET(H)$ is NP-complete by an obvious reduction from the undirected case [3].

Next we employ an indicator I with four vertices, i, j, k, ℓ and five edges $ki, i\ell, kj, j\ell, k\ell$. Consider all possible homomorphisms of I to H. The edge $k\ell$ may be mapped to an edge between consecutive vertices a_t and a_{t+1} (either k to a_t and ℓ to a_{t+1}, or conversely, depending on whether the edge in H is forward or double, or backward). In this case i can map either to a_t or to a_{t+1} and the same applies to j. Thus yields for H^* all double edges joining a_t with a_{t+1}. The edge $k\ell$ may also be mapped to some loop a_ta_t: in this case i and j could also be mapped to a_{t-1}, a_{t+1}, as long as H contains consecutive double edges on $a_{t-1}a_t$ and a_ta_{t-1}.

We conclude that as long as H does not contain consecutive double edges, H^* is a cycle of double edges of length at least four, and hence $RET(H^*)$ is NP-complete and so is $RET(H)$. Otherwise, assume that a_1a_2 and a_2a_3 are double edges. In this case, the associated bipartite graph $B(H)$ always contains an induced cycle of length at least six: consider the edges $a_1a_2', a_2'a_3$, and a shortest path from a_3 or a_3' to a_1 or a_1'. (A similar argument for undirected graphs is given in more detail in [15], Proposition 4). Thus $RET(B(H))$ is NP-complete by [4], and hence so is $RET(H)$. □

Corollary 2.5 *For reflexive digraphs H, the problems $SubRET(H)$, $CL - HOM(H)$, and $L - HOM(H)$ are all NP-complete, unless both $U(H)$ and $S(H)$ are chordal graphs.*

We note that $SubRET(H)$ may be NP-complete even if both $U(H)$ and $S(H)$ are chordal - as in the case of the reflexive directed three-cycle C_3 [6].

We also note that the problem $RET(H)$ may be polynomial time solvable even when $U(H)$ and $S(H)$ are both non-chordal: for instance when H is a reflexive *wheel*, which is the digraph obtained from a symmetric cycle by adjoining one vertex dominating all other vertices. It is easy to see that G admits a retraction to H if and only if the removal of all vertices of indegree zero from G results in a digraph that retracts to the symmetric cycle. It follows from [3, 8] that dichotomy for $RET(H)$ for reflexive digraphs H would imply dichotomy for all of constraint satisfaction problems.

3 Interval graphs and digraphs

It follows from the previous section that induced cycles of length at least four, as well as directed cycles of length three, are structures whose presence in a reflexive digraph H causes the NP-completeness of the list homomorphism problem $L - HOM(H)$. In this section we identify additional structures with this property.

An *asteroidal triple* in a graph H is a triple of vertices $0, 1, 2$ and paths $P(0, 1), P(0, 2), P(1, 2)$ (where $P(i, j)$ joins vertices i and j), such that each vertex i from $0, 1, 2$ has no neighbours on the path joining the other two vertices. It is known that a graph is an interval graph if and only if it is chordal and has no asteroidal triple [19]. If a reflexive graph H contains an asteroidal triple, Then the problem $L - HOM(H)$ is NP-complete[3]. A similar result holds about reflexive digraphs; in the spirit of our convention, an asteroidal triple in H is an asteroidal triple in $U(H)$.

Theorem 3.1 *Let H be a reflexive digraph. If $U(H)$ contains an asteroidal triple, then $L - HOM(H)$ is NP-complete.*

The theorem can be derived from [4] by passing through the associated bipartite graph $B(H)$ as above. However, for future needs, we develop direct tools that will be used in the rest of the paper.

Proof. Suppose $U(H)$ contains an asteroidal triple with vertices $0, 1, 2$ and paths $P(0,1), P(0,2), P(1,2)$.

We first recall gadgets called choosers from [3, 18], as discussed in [17]. We state the definition in a slightly more general form, and apply it to digraphs. Let i, j be distinct vertices from $0, 1, 2$ and let I, J be subsets of $\{0, 1, 2\}$. A *chooser* $Ch(i, I; j, J)$ is a digraph P with specified vertices a and b, and with lists $L(p) \subseteq V(H)$, for $p \in V(P)$, such that any list homomorphism f of P to H has $f(a) = i$ and $f(b) \in I$ or $f(a) = j$ and $f(b) \in J$; and for any $i' \in I$ and $j' \in J$ there is a list homomorphism f of P to H with $f(a) = i$ and $f(b) = i'$ and a list homomorphism g of P to H with $g(a) = j$ and $g(b) = j'$.

It is shown in [3], as explained in [17] page 174-5, that if there exist choosers $Ch(i, \{i, k\}; j, \{j, k\})$ and $Ch(i, \{i\}; j, \{k\})$, for any permutation ijk of $0, 1, 2$, then $L - HOM(H)$ is NP-complete. (Those proofs are stated in terms of undirected graphs H and choosers Ch that are paths, but they apply verbatim to arbitrary digraph choosers Ch as defined here.)

These choosers will be constructed from simpler building blocks which we call separators. A *separator* $G(i), i = 0, 1, 2$, is a digraph with two specified vertices u, v and lists $L(t), t \in V(G(i))$, such that

- every list homomorphism of $G(i)$ to H with respect to the lists L maps both u, v to i or maps neither of u, v to i, and
- for any pair of values x, y from $0, 1, 2$ in which neither or both values x, y are equal to i, there is a list homomorphism of $G(i)$ with respect to the lists L, mapping u to x and v to y.

The proof will be completed by the following two lemmas. □

Lemma 3.2 *If there exists a separator $G(i)$ for each $i = 0, 1, 2$, then the problem $L - HOM(H)$ is NP-complete.*

Proof. The separators can be used to construct the choosers as follows: $Ch(i, \{i\}; j, \{k\})$ is formed from $G(i)$ by setting $a = u$ and modifying its list to $L(u) = \{i, j\}$, and by setting $b = v$ and modifying its list to $L(v) = \{i, k\}$. To form $Ch(i, \{i, k\}; j, \{j, k\})$, we take four vertices a, b, c, d, and place one copy of $G(i)$ between a and c (identifying a with u and c with v), and another copy of $G(i)$ between b and d (identifying in a similar manner), as well as a copy of $G(j)$ between c and b and another copy of $G(j)$ between d and a. It is easy to check that the resulting digraph satisfies the conditions for a chooser $Ch(i, \{i, k\}; j, \{j, k\})$ with the specified vertices a and b. □

Lemma 3.3 *If the $U(H)$ has an asteroidal triple, then H has separators $G(i), i = 0, 1, 2$.*

Proof. Suppose H has n vertices, and $U(H)$ has an asteroidal triple $0, 1, 2$. The separator $G(i)$ will be an oriented path of length $2n$, with alternating forward and backward edges. The lists of the two end vertices of the path $G(i)$ are $\{0, 1, 2\}$. All other vertices of $G(i)$ have lists consisting of i, together with all the vertices on the path $P(j, k)$ in H (from the definition of an asteroidal triple). Note that the length of the path $G(i)$ and the orientation of its edges ensure that it admits a homomorphism (without considering the lists) that maps u and v to any two vertices of H (recall that every vertex has a loop). It follows from the definition of an asteroidal triple that any list homomorphism of $G(i)$ to H maps both u and v to i, or neither of u, v to i; and moreover, that there are list homomorphisms of $G(i)$ to H mapping both u and v to i and both to j, k in any prescribed combination, i.e., that $G(i)$ is a separator. □

The reader will have noticed that the three vertices $0, 1, 2$, together with the three separators $G(0), G(1), G(2)$, form a weak version of an asteroidal triple in the digraph H. Note that the separators $G(i)$ are not subgraphs of H; nevertheless, in all our constructions they will be of size polynomial in H.

Corollary 3.4 *For reflexive digraphs H, the problem $L - HOM(H)$ is NP-complete, unless both $U(H)$ and $S(H)$ are interval graphs.*

We now introduce a variant of interval digraphs, better suited for list homomorphism problems. Let H be a reflexive digraph. We say that an interval pair representation $I_v, J_v, v \in V(H)$ of H is *adjusted*, if the left endpoint of I_v equals the left endpoint of J_v, for all $v \in V(H)$. We say that H is an *adjusted interval digraph* if it admits an adjusted interval pair representation. It turns out that if H is an adjusted interval digraph, then there is a polynomial time algorithm for the list homomorphism problem $L-HOM(H)$. This is best seen by relating adjusted interval pair representations to certain vertex orderings.

A min *ordering* of a digraph H is an ordering of the vertices of H such that whenever xy and $x'y'$ are edges of H, then $\min(x,x')\min(y,y')$ is also an edge of H. (A min ordering is also called an *X-underbar enumeration* [12, 17]). For reflexive digraphs, a min ordering can be described in a simpler language.

Lemma 3.5 *Let H be a reflexive digraph. Then an ordering $<$ of $V(H)$ is a* min *ordering if and only if for any three vertices $i < j < k$ we have*

- *$ik \in E(H)$ implies $ij \in E(H)$, and*
- *$ki \in E(H)$ implies $ji \in E(H)$.*

Proof. The necessity of the two properties follows by taking the arc ik (respectively ki) and the loop at j. To see the sufficiency, consider edges $xy, x'y'$ of H and assume without loss of generality that $x < x', y' < y$; thus $\min(x,x')\min(y,y') = xy'$. If $x = y'$, then xy' is an edge since H is reflexive. If $x < y'$, then xy' is an edge because of the triple $x < y' < y$. If $y' < x$, then xy' is an edge because of the triple $y' < x < x'$. □

Corollary 3.6 *Let H be a reflexive digraph. An ordering of the vertices of H is a* min *ordering if and only if for each vertex v the vertices that follow v in the ordering consist of*

1. *first, all vertices that are adjacent to v by double edges,*
2. *second, all vertices that are adjacent to v by single edges, either all forward or all backward, and*
3. *last, all vertices that have no edges to or from v.*

Of course, any of the three groups could be empty. Note that, in particular, in a min ordering of H it cannot be the case that a vertex v has both forward and backward edges towards vertices that follow it in the ordering.

We now derive the connection between adjusted interval representations and min orderings.

Theorem 3.7 *A reflexive digraph is an adjusted interval digraph if and only if it admits a* min *ordering.*

Proof. Given a min ordering, we can arrange the points l_v in the same order as they appear in the min ordering, and select intervals I_v and J_v as follows. If v has no forward edges towards later vertices, we end the interval I_v at the last vertex w such that vw is a double edge, and end the interval J_v at the last vertex w such that vw is a backward edge. If v has no backward edges towards later vertices, we end the interval J_v at the last vertex w such that vw is a double edge, and end the interval I_v at the last vertex w such that vw is a forward edge. Conversely, given an adjusted interval pair representation $I_v, J_v, v \in V(H)$ we obtain a min ordering of H according to the left endpoints of the intervals. □

Corollary 3.8 *If H is a reflexive adjusted interval digraph, then $L-HOM(H)$ is solvable in polynomial time.*

Proof. If H admits a min ordering, then $L - HOM(H)$ is polynomial time solvable by [12], cf. [17]. (In fact, the problem is of *width one* in the terminology of [8]). □

Conjecture 3.9 *Let H be a reflexive digraph. If H is an adjusted interval digraph, then $L - HOM(H)$ is polynomial time solvable; otherwise, $L - HOM(H)$ is NP-complete.*

As noted above, the first claim is known to hold. Proving the second claim would be facilitated by having a forbidden substructure characterization of adjusted interval digraphs; this was the case for undirected graphs [3, 4, 5]. We believe such a characterization may exist, and present towards this purpose a digraph analogue of asteroidal triples.

A *di-asteroid* in H is a set S of vertices such that for each ordering $<$ of S there exist vertices $i < j < k$ in S and a path P in H from i to k which *avoids* j. The path $P : i = x_0, x_1, \ldots, x_p = k$ avoids j if the following is true:

- if $x_t x_{t+1}$ is a forward arc then $x_t j \notin E(H)$
- if $x_t x_{t+1}$ is a backward arc then $j x_t \notin E(H)$.

Proposition 3.10 *If H has a di-asteroid, then it is not an adjusted interval digraph.*

Proof. Suppose H is an adjusted interval digraph, and consider the min ordering $<$ of its vertices. Then for any $i < j < k$ and any path P from i to k there must be an edge $x_t x_{t+1}$ such that $x_t < j < x_{t+1}$; this contradicts Lemma 3.5. □

In this generality, H is an adjusted interval digraph if and only if it does not have a di-asteroid, since we can take $S = V(H)$ and the paths P consisting of a single edge. Of course, this is not a useful reformulation of the definition. However, we believe there is a useful set of obstructions, which are di-asteroids of bounded size. As far as we know it is even possible that it suffices to look for di-asteroids of size up to four.

Conjecture 3.11 *A digraph H is an adjusted interval digraph if and only if it does not have a di-asteroid S with $|S| \leq 4$.*

It is worth noting that unlike the case of interval graphs, where asteroidal triples as well as induced cycles must be forbidden, the conjecture would characterize adjusted interval digraphs by the absence of just asteroids (of size three or four). For instance, symmetric four and five cycles contain di-asteroids of size four, and longer cycles contain di-asteroids of size three.

We verify the conjectures in several special cases, focusing on trees and complete graphs - the building blocks of interval graphs.

We now observe that di-asteroids of size three generalize asteroidal triples.

Proposition 3.12 *If $U(H)$ has an asteroidal triple, then H has a di-asteroid of size three.*

In fact, the proof of Theorem 3.1 can be extended to yield the following fact.

Proposition 3.13 *If H has a di-asteroid of size three, then $L-HOM(H)$ is NP-complete.*

Proof. Suppose $S = \{0, 1, 2\}$. The separators $G(i), i = 1, 2, 3$, are obtained in a similar way as in the proof of Theorem 3.1. Specifically, $G(i)$ is obtained by taking two paths from the definition of a di-asteroid - one for the order $j < i < k$ and one for the order $k < i < j$, identifying the first vertex of one with the last vertex of the other. □

We observe that if an adjusted interval digraph H has no double edges, then we can replace one of the intervals I_v, J_v by a single vertex (see the proof of Theorem 3.7). The resulting characterization is even closer to interval graphs. (See also [22].) For convenience we say that an interval I *follows* an interval I' if the left endpoint of I is greater than or equal to the left endpoint of I'.

Theorem 3.14 *Suppose H is a reflexive antisymmetric digraph. Then H admits a* min *ordering if and only if the underlying graph $U(H)$ has an interval representation $I_v, v \in V(H)$, in which the following property holds in the digraph H (with the vertices now being viewed as the intervals):*

- *either $vx \in E(H)$ for all x such that I_x intersects and follows I_v,*
- *or $xv \in E(H)$ for all x such that I_x intersects and follows I_v.*

We remark that Benoit Larose has shown that a reflexive antisymmetric digraph H either has a min ordering or the problem $L - HOM(H)$ is NP-complete. Thus by Theorem 3.7, we conclude that Conjecture 3.9 is true for antisymmetric digraphs.

4 Trees

Here we verify our conjectures for reflexive digraphs H for which $U(H)$ is a tree. It is well known [9] that a tree is an interval graph if and only if it is a caterpillar. (A tree H is a *caterpillar* if the removal all leaves results in a path P.) Let $S(x)$ denote the set of leaves of H adjacent to the vertex $x \in P$. As usual, we refer to H as a tree, or star, etc., to mean that $U(H)$ is a tree, or star, etc., respectively.

If H is a star, we shall define H to be a *good caterpillar*, if it does not contain, as induced subgraph, the tree T_2 depicted below. If H is not a star, we define it to be a *good caterpillar* if it has a longest path $v_0, v_1, \ldots,$ v_k, v_{k+1} satisfying the following conditions for all i. (Note that $v_1, v_2, \ldots, v_k$ is the path P, and that $v_0 \in S(v_1), v_{k+1} \in S(v_k)$.)

1. If $v_iv_{i+1} \in E(H)$, then $v_iv \in E(H)$ for all $v \in S(v_i) - v_{i-1}$.

2. If $v_{i+1}v_i \in E(H)$, then $vv_i \in E(H)$ for all $v \in S(v_i) - v_{i-1}$.

Note that if v_iv_{i+1} is a double edge then so are all $v_iv, v \in S(v_i) - v_{i-1}$. Observe that there are no restrictions on v_0, other than those arising from the restrictions on v_1. Indeed, all edges v_1v for $v \in S(v_1) - v_0$ must follow the direction of the edge v_1v_2 (forward, backward, or double) - with the possible exception of a single vertex v, which must be the vertex v_0. Thus such a v_0 can be chosen if and only if the restrictions on v_1 have at most one exception. Similarly, there are no restrictions on v_{k+1}, other than those arising from the restrictions on v_k. All edges v_kv for $v \in S(v_k)$ must follow the direction of the edge v_kv_{k+1}. It is easy to see that such a v_{k+1} can be chosen if and only if between v_k and $S(v_k)$ there does not exist at the same time a single forward and a single backward edge. Finally, we note that the exceptional case, when H is a star, also conforms to the general definition; we have chosen to state it separately only for convenience.

Theorem 4.1 *Let H be a reflexive digraph that is a tree. Then the following statements are equivalent.*

1. *H is a good caterpillar*

2. *H has a* min *ordering*

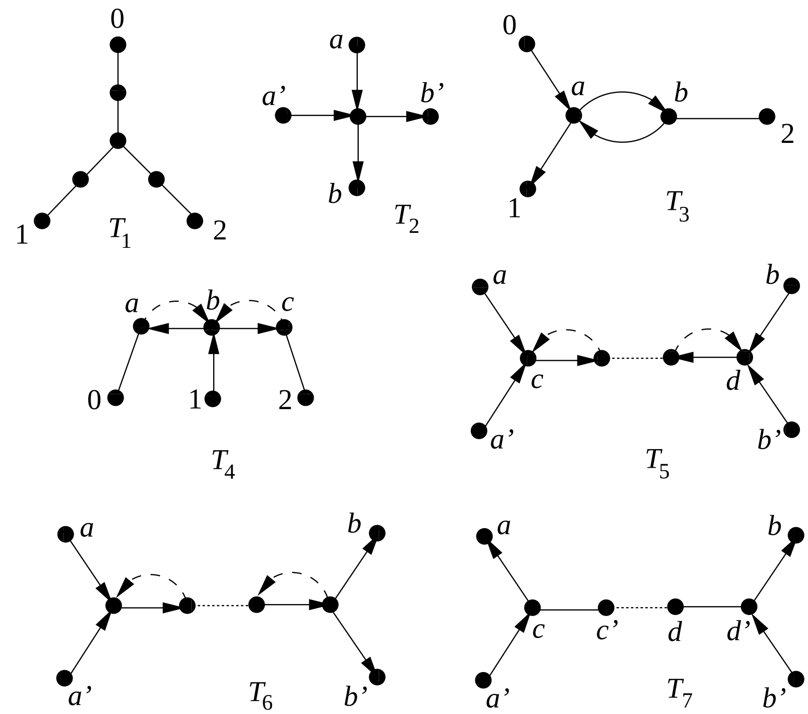
3. *H does not contain (as an induced subgraph) any of the trees $T_1, \ldots T_7$ or their reverses.*

Proof. The edges in the trees $T_1, \ldots T_7$ that are not oriented can be forward, backward, or double; the dashed edges are optional.

We shall show that 1 implies 2, 2 implies 3, and 3 implies 1. Indeed, 1 implies 2, as a good caterpillar can be ordered starting from v_0 and proceding to $v_1, v_2, \ldots, v_k$, with listing the double edges of $S(v_i) - v_{i-1}$ first, as suggested by Corollary 3.6. The definition of a good caterpillar ensures that the listing for $S(v_i) - v_{i-1}$ can be chosen to end with v_{i+1}.

It is easy to check that none of the forbidden subtrees allows a min ordering, thus 2 implies 3. In fact, we can use Theorem 3.1 and Proposition 3.10: the tree T_1 contains the asteroidal triple $0, 1, 2$, the tree T_2 contains the di-asteroid a, a', b, b', the tree T_3 contains the di-asteroid $0, 1, 2$, and the remaining trees contain the di-asteroid a, a', b, b'.

It remains to show that 3 implies 1. Thus suppose H is a reflexive tree which does not contain any of $T_1 - T_7$ or their reverses. Since H does not



contain T_1 it is a caterpillar. If H is a star, the conclusion now follows. Thus assume H is not a star: when all leaves of H are removed we obtain a path P, say $P = p, r, s, \ldots, y, z$. We will prove that one of p, z can be chosen as v_1 and the other as v_k. Suppose first that p cannot be chosen to satisfy the condition for v_1. Then in $S(p)$ there must be two vertices v, v' such that the edges pv, pv' do not follow the direction of the edge pr on P. If pr is a double edge, this means that pv, pv' are single edges. Since H does not contain T_3, both are forward (or both backward) edges. This implies that all edges $pv, v \in S(p)$ follow the direction of pr, and thus p can be chosen to satisfy the condition for v_k. Similarly, if pr is a single (forward or backward edge), p can be chosen as v_k, since H does not contain T_2. Therefore, each of p, z satisfies the condition for v_1 or for v_k. Suppose next that neither p nor z satisfy the condition for v_1. Then each contains two single edges whose direction does not follow the direction of pr; this contradicts the fact that H does not contain T_5 and T_6 or their reverses. Similarly, the absence of T_7 implies that at least one of p, z satisfies the condition for v_k. The absence of T_4 (and its reverse) implies that each intermediate vertex $r, s, \ldots, y$ of P satisfies the condition for v_i if its left or its right neighbour on P plays the role of v_{i+1}. Finally, if one vertex of P requires its left neighbour, while another requires its right neighbour, we again obtain a contradiction as above with the fact that H does not contain the trees T_5, T_6, T_7. □

Corollary 4.2 *Let H be a reflexive digraph that is a tree.*

If H is a good caterpillar, then $L-HOM(H)$ is polynomial time solvable. Otherwise, $L - HOM(H)$ is NP-complete.

Proof. If H is a good caterpillar, the theorem implies that it has a min ordering and hence $L - HOM(H)$ is polynomial time solvable. Otherwise, the theorem implies that H contains $T_1, T_2, \ldots,$ or T_7.

If H contains T_1, it has an asteroidal triple and hence $L - HOM(H)$ is NP-complete by Theorem 3.1.

If H contains T_2, then we shall apply Lemma 2.2. Consider the indicator I consisting of three vertices i, c, j and two edges ic, cj, with the lists $L(i) = L(j) = \{a, a', b, b'\}$, $L(c) = V(H)$. It is clear that H^* is a reflexive digraph that is a cycle with four vertices. Thus $L - HOM(H^*)$ is NP-complete by Theorem 2.4, and $L - HOM(H)$ is NP-complete by Lemma 2.2.

If H contains T_3 then consider the three vertices $0, 1, 2$ of T_3. We shall prove that $L - HOM(H)$ is NP-complete using Lemma 3.2. Indeed, since there is a path joining $0, 1$ that avoids the neighbours of 2, the separator $G(2)$ is constructed as in Theorem 3.1. To construct $G(1)$, we take a path that begins with a forward and then a double edge, followed by a sufficiently long alternating sequence of forward and backward arcs, and ending with a double edge followed by a backward arc. The lists will be $\{0, 1, 2\}$ everywhere except a will be added to the lists of the second and second to last vertex and b will be added to the third and third to last vertex. This pattern of edges and lists ensures that there is a list homomorphism mapping the first vertex of $G(1)$ to 0 and the last vertex to 2 and conversely, while if the first or last vertex of $G(1)$ is mapped to 1, the entire path must map to 1. The path $G(0)$ is constructed similarly. By Lemma 3.2, $L - HOM(H)$ is NP-complete.

If H contains T_4, we proceed similarly, Only $G(1)$ requires an explanation: it is enough to take a sufficiently long path of alternating forward and backward edges with a middle vertex t of indegree zero, and assign the lists $\{0, 1, 2\}$ to the end vertices, the lists $\{0, 1, 2, a, c\}$ to all inner vertices except t, and the list $\{0, 1, 2, a, b, c\}$ for the special vertex t. It is again easy to check that this pattern of forward and backward edges, together with the lists, ensure the required properties for the separator $G(1)$.

If H contains T_5, we shall again use Lemma 2.2. The indicator will be a path I from i to j identical to the path $a, c, \ldots, d, b$ in T_5. the lists are

$L(i) = L(j) = \{a, a', b, b'\}$ and otherwise $L(x) = \{x, a, a', b, b'\}$. It is easy to check that H^* is a reflexive cycle with four vertices. The proof for T_6 is similar.

Consider now the last tree T_7. If the edge cc' or dd' is double, T_7 contains T_3 and hence we are done. Thus we shall assume that $c'c, dd'$ are forward edges. (By relabeling we obtain the case when they are both backward edges; the case when one is forward and the other backwards is different, but the proof is similar.) We again proceed to use Lemma 2.2. The indicator will be a path I from i to j consisting of a path from i to a middle vertex t identical to the path $a, c, c', \ldots, d', b'$ in T_7, followed by a path from t to j identical to the path $a', d', d, \ldots, c, b$ in T_7. The lists are $L(i) = L(t) = L(j) = \{a, a', b, b'\}$ and $L(x) = V(H)$ otherwise. It is easy to check that H^* is the reflexive cycle with edges $ab, ab', a'b, a'b'$. (The path from i to t ensures the presence of the edges $ab, a'b$ and the path from t to j ensures the presence of the edges $ab', a'b'$.) □

To complete the picture, we now show that when H is a reflexive digraph such that $U(H)$ is a tree, then the problem $SubRET(H)$ is polynomial time solvable.

Theorem 4.3 *If H is a reflexive digraph such that $U(H)$ is a tree, then $CL-HOM(H)$, $SubRET(H)$, and $RET(H)$, are all polynomial time solvable.*

Proof. Clearly it suffices to prove the claim for $CL-HOM(H)$. A *majority function* on H is a a ternary function g on the vertices of H satisfying the following properties:

1. $g(x, x, y) = g(x, y, x) = g(y, x, x) = x$, for any two vertices x, y in H;
2. $g(x, y, z)g(x', y', z')$ is an edge in H, for any three edges xx', yy', zz' in H.

Feder and Vardi [8] have shown that if H admits a majority function then the problem $HOM(H)$ is polynomial time solvable. It follows from their proof that the problem $CL-HOM(H)$ is also polynomial time solvable (and of strict width two, in the terminology of [8]) as long as the majority function satisfies the following additional property:

- $g(x, y, z)$ belongs to every set S of vertices containing x, y, z which induces a (weakly) connected subgraph of H, for any three vertices x, y, z in H.

A reflexive tree H admits such a majority function. Indeed, in the underlying undirected tree of H, any three vertices x, y, z admit a unique vertex t lying on each of the paths from x to y, from x to z, and from y to z [20]. We define $g(x, y, z) = t$; it can be verified that this function g is a majority function and satisfies the additional property. □

We note that the result for $RET(H)$ also follows from [13, 21], cf. [17].

5 Semi-complete digraphs

A digraph H is *semi-complete* if its underlying graph $U(H)$ is complete. In particular, if H is a semi-complete digraph, then $U(H)$ is an interval graph. If $S(H)$ is not an interval graph, then $L - HOM(H)$ is NP-complete by Corollary 3.4; similarly, if H contains induced C_3, then $L - HOM(H)$ is NP-complete by [6]. We say that a reflexive semi-complete digraph H is *admissible* if it contains no induced C_3 and $S(H)$ is an interval graph.

Let R be the reflexive digraph with vertices $0, 1, 2$ (each with a loop) and with the additional edges $01, 10, 12, 20$. We first consider R-free digraphs, i.e., digraphs without induced copy of R.

Theorem 5.1 *Suppose H is an R-free reflexive semi-complete digraph. If H is admissible, then $L - HOM(H)$ is polynomial time solvable. Otherwise it is NP-complete.*

Proof. The NP-completeness is justified above. The polynomial algorithm will follow from the next lemma, see Corollary 5.3. □

An interval pair representation $I_v, J_v, v \in V(H)$ is called *special* if each interval I_v extends from its left endpoint (shared by J_v) to infinity. If intervals J_v and J_w intersect then vw is a double edge, while if intervals J_v and J_w do not intersect, with J_v to the left of J_w, then vw is a forward edge.

Lemma 5.2 *Let H be an R-free admissible digraph. Then there exists a special interval pair representation of H.*

Proof. Note that a special interval pair representation is completely described by the intervals $J_v, v \in V(H)$.

In the interval graph $S(H)$ there is a vertex u whose neighbors form a clique Q. Let H' be the reflexive digraph obtained from H by removing the vertex u, and assume inductively that H' admits a special interval pair representation. Let $V \subseteq V(H)$ be the set of vertices $v \neq u$ such that $vu \in E(H)$ is a single edge, and let $W \subseteq V(H)$ be the set of vertices $w \neq u$ such that $uw \in E(H)$ is a single edge. Then for $v \in V$ and $w \in W$, the edge vw cannot be a double edge, or else v, u, w would induce a copy of R. Furthermore the edge vw cannot be backward, since otherwise v, u, w would form a reflexive directed cycle. Therefore vw is a forward edge, i.e., an edge of H.

It follows that all the intervals J_v for v in V are to the left of the intervals J_w for w in W. We may thus insert an interval J_u for u between the intervals for V and the intervals for W, satisfying the conditions for a special interval pair representation as far as the directed edges vu and uw with v in V and w in W are concerned.

It remains to ensure that the intervals J_q for q in the clique Q intersect the interval J_u. Suppose to the contrary that some such J_q is say to the left of J_u. There cannot be an interval J_v with v in V between J_q and J_u that does not intersect either J_q or J_u, since otherwise q, v, u would form an R. Furthermore, any interval $J_{q'}$ with q' in Q intersects J_q, since Q is a clique in $S(H)$. We may thus extend J_q to the right until J_q meets J_u without changing any of the intersections other than making J_q meet J_u. Similarly, an interval J_q with q in Q to the right of J_u may be extended to the left until J_q meets J_u without changing any of the intersections other than making J_q meet J_u. Thus J_u now meets all intervals J_q with q in Q, providing the desired special interval pair representation for H. □

Corollary 5.3 *If H is an R-free admissible digraph, then H is an adjusted interval digraph and $L-HOM(H)$ is polynomial time solvable.*

Note that we have now verified Conjecture 3.9 for R-free reflexive semi-complete digraphs.

Let R_1 be a copy of R in H on vertices v_1, v_2, v_3, with edges $v_1v_2, v_2v_3, v_1v_3, v_3v_1$, and let R_2 be a copy of R in H on vertices v_4, v_5, v_6, with edges $v_4v_5, v_5v_6, v_4v_6, v_6v_4$. We shall say that R_1 and R_2 are *badly matched* in H if the vertices v_1, v_2, v_5, v_6 are pairwise distinct and furthermore all v_iv_j with $1 \leq i \leq 3$ and $4 \leq j \leq 6$ are edges of H, with the possible exception of v_1v_6 and v_3v_4. It is not hard to verify that if H contains two badly matched copies of R, then it contains a di-asteroid of size four. However, we are more interested in the following fact.

Lemma 5.4 *If H contains two badly matched copies of R, then $L-HOM(H)$ is NP-complete.*

Proof. Let I, i, j be an indicator with vertices i, j, z, t with lists $L(i) = L(j) = \{v_1, v_2, v_5, v_6\}$, $L(z) = \{v_2, v_3\}$, and $L(t) = \{v_4, v_5\}$, and with edges zi, zj and it, jt. Then H^* contains the reflexive symmetric cycle of length four $v_1v_5v_2v_6v_1$, and hence both $L - HOM(H^*)$ and $L - HOM(H)$ are NP-complete, by Theorem 2.4 and Lemma 2.2 respectively. □

We now consider forbidding another reflexive three-vertex digraph R' with vertices $0, 1, 2$ (each with a loop) and additional edges $01, 10, 12, 20, 02$. We shall again consider R'-free digraphs. It is easy to see that a semi-complete digraph is R'-free if and only if $S(H)$ is a disjoint union of cliques (with no other edges).

Theorem 5.5 *Suppose H is an R'-free reflexive semi-complete digraph.*

If H contains an induced C_3 or two badly matched copies of R, then $L - HOM(H)$ is NP-complete.

Otherwise, H is an adjusted interval digraph and $L - HOM(H)$ is polynomial time solvable.

Proof. Suppose the single edges of H contain a directed cyle. Let C be a shortest cycle of single edges of H. Then every pair of vertices in $V(C)$ is adjacent either by an edge in C or by a double edge. If $V(C)$ has three vertices, then $L - HOM(H)$ is NP-complete because $L - HOM(C_3)$ is NP-complete [6]. If $V(C)$ has four vertices with C given by $w_1w_2w_3w_4w_1$, then there are badly matched copies of R on $U = \{w_1, w_2, w_3\}$ and $V = \{w_2, w_3, w_4\}$, so $L - HOM(H)$ is NP-complete by the above lemma. If $V(C)$ has at least five vertices, then the symmetric edges on $V(C)$ do not form a union of cliques. (In fact they form a graph that is not chordal, hence $L - HOM(H)$ is NP-complete).

We may thus assume that the single edges of H form an acyclic digraph; let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of H so that every single edge of H is of the form v_iv_j with $i < j$. Suppose that H is a minimal counterexample to the Theorem. Recall that the double edges of H form a union of cliques.

If the clique L of double edges containing v_1 has precisely the vertices v_i for $1 \leq i \leq l$ for some l, then a min ordering of H is obtained by listing first the vertices of L, followed by a min ordering of the remaining vertices forming a graph H' which has such an ordering by the minimality of H. Similarly, if the clique of double edges M containing v_n has precisely the vertices v_i for $m \leq i \leq n$ for some m, then a min ordering again lists first the vertices in M. If the clique of double edges L containing v_1 has precisely the vertices v_i for $1 \leq i \leq l$ and for $m \leq i \leq n$, then a min ordering again lists first the vertices in L.

The preceding three cases exclude several situations. First there is the case where the clique of double edges L contains v_1 and v_n, and also contains some additional v_j, but does not contain either v_i or v_k with $1 < i < j < k < n$. Then we obtain badly matched copies of R on $U = \{v_1, v_i, v_j\}$ and $V = \{v_j, v_k, v_n\}$. The remaining case has a clique of double edges L containing v_1 and a clique of double edges M containing v_n, so that either L contains a vertex v_j and M contains a vertex v_i with $1 < i < j < n$, in which case we obtain badly matched copies of R with $U = \{v_1, v_i, v_j\}$ and $V = \{v_i, v_j, v_n\}$; or L contains a vertex v_j and does not contain a vertex v_i and M contains a vertex v_k and does not contain a vertex v_r with $1 < i < j < k < r < n$, in which case we obtain badly matched copies of R on $U = \{v_1, v_i, v_j\}$ and $V = \{v_k, v_r, v_n\}$. NP-completeness follows by Lemma 5.4. □

Corollary 5.6 *Let H be a reflexive semi-complete digraph, and let H' be obtained from H by replacing each connected component of $S(H)$ by a symmetric clique. If H' is not an adjusted interval digraph, then $L - HOM(H)$ is NP-complete.*

Proof. Let L and M be two connected components of the symmetric part of H. We claim that if L has a vertex u and two vertices v, w in M with edges vu, uw, and M has a vertex u' and two vertices v', w' in M with edges $v'u', u'w'$, then H has two badly matched copies of R and the claim follows. By a connectivity argument, we may assume v, w are adjacent by a double edge. If u'' is adjacent to u by a double edge, then we must also have edges $vu'', u''w$, otherwise u, u', v, w induce a badly matched pair. It follows that for all u'' in L we have edges $vu'', u''w$. If v'' is adjacent to v or to w by a double edge, and we have both edges $v''x, yv''$ for some x, y adjacent by double edges in L, then v, v'', x, y or w, v'', x, y again induce two badly matched copies of R. Therefore, for each v'' in M, either we have all edges $v''x$ or all edges xv'' for x in L, proving the claim.

We now define an indicator I, i, j with $i = x, j = y$ and additional vertices t, z as well all vertices on a path from x to z with $|V(H)|$ double edges and on a path from t to y with $|V(H)|$ double edges. In addition to the edges on the paths, I also includes the edges xt, zy. By the above claim, $H^* = H'$, completing the proof. □

All evidence sofar points to the following possibility.

Conjecture 5.7 *For semi-complete reflexive digraphs H, the problem $L-HOM(H)$ is polynomial time solvable when H is admissible and does not contain two badly matched copies of R, and it is NP-complete otherwise.*

The NP-completeness has already been proved above; for the polynomial algorithms we have covered the case of no copies of R at all, so it remains to consider reflexive digraphs which have copies of R but none are badly matched. If such digraphs could be shown to be adjusted interval digraphs we would be done.

We now turn to the problems $SubRET(H)$. Theorem 5.1 has the following corollary.

Corollary 5.8 *Suppose H is an R-free reflexive semi-complete digraph.*

If H is admissible, then $SubRET(H)$ is polynomial time solvable. Otherwise $SubRET(H)$ is NP-complete.

Proof. It only remains to prove the NP-completeness claim, for digraphs H such that $S(H)$ is chordal. The result follows from the next lemma. □

Lemma 5.9 *Let G be a chordal graph.*

Then $G = S(H)$ for some R-free semi-complete reflexive digraph H if and only if G is an interval graph.

Proof. For the necessity, if G is not an interval graph, then G has an asteroidal triple $0, 1, 2$. We may assume without loss of generality that the edge 01 in H is oriented from 0 to 1. Let p be the path from 1 to 2 in G not containing any neighbors of 2. Then for successive vertices u on p, the edge $0p$ in H is oriented from 0 to p, since a change in orientation for consecutive vertices u and v on p would make the triangle $0uv$ in H isomorphic to R. Therefore the edge 02 in H is oriented from 0 to 2.

We have thus shown that the edges 01 and 02 must both be oriented from 0 or both towards 0 in H. A similar argument shows that the edges 10 and 12 must both be oriented from 1 or both toward 1 in H, and that the edges 20 and 21 must both be oriented from 2 or both toward 2 in H. Thus if say 01 is oriented from 0 to 1, then 02 is oriented from 0 to 2, so 12 is oriented from 1 to 2, so 10 is oriented from 1 to 0, a contradition.

For the sufficiency, if G is an interval graph, let $I_v, v \in V(G)$ be an interval representation of G. Orient an edge uv of H not in G from u to v if and only if I_u is to the left of I_v, where these intervals do not intersect. If three vertices u, v, w in H form an R with edges uv, vw, uw, wu, then the intervals for u and w must intersect, but I_u is to the left of I_v, and I_v is to the left of I_w, so I_u and I_w do not intersect, a contradiction. □

6 Partial orders

The reflexive digraph H is a *partial order* if the relation of adjacency is transitive and antisymmetric. Feder and Vardi [8] showed that obtaining a dichotomy for $RET(H)$ for partial orders H is equivalent to obtaining such a dichotomy for all constraint satisfaction problems. We show that if H is a partial order, the problem $SubRET(H)$ is polynomial if the underlying graph K is chordal, and is NP-complete otherwise.

Let T be the reflexive digraph with four vertices x, y, z, t, (with loops) and additional edges xy, xt, yz, zt, yt. Let T' be obtained from T by reversing the direction of the edges.

Theorem 6.1 *Let H be a reflexive antisymmetric digraph not containing induced T or T'.*

If $U(H)$ is chordal and H is C_3-free, then $SubRET(H)$ is polynomial time solvable.

Otherwise, $SubRET(H)$ is NP-complete.

Corollary 6.2 *Let H be a partial order.*

If H is chordal, then $SubRET(H)$ is polynomial. Otherwise, $SubRET(H)$ is NP-complete.

Proof. We have already shown the NP-completeness.

Since the property of H of having a chordal underlying graph is preserved under induced subgraphs, it suffices to show that the problem $RET(H)$ is polynomial.

Since H has a chordal underlying graph and does not contain C_2, C_3 as induced sugraphs, it follows that the only cycles of H are loops. We shall in addition to the single vertex lists and the all-vertex lists allow also lists S such that S is contained in a clique of the underlying graph of H and if $y, t \in S$ with yz, zt, yt edges of H then $z \in S$.

Since H is chordal, it contains a vertex a whose neighbors induce a clique. Let B be the set of vertices in H other than a that have an edge to a, and let C be the set of vertices in H other than a that have an edge from a. If B is nonempty, let b be the element of B such that every vertex in B has an edge to b; if C is nonempty, let c be the element of C such that every vertex in C has an edge from c.

For any two vertices in an instance that must map to a, all vertices on a directed path joining these two vertices must also map to a. If B is empty, then all vertices that have a directed path to a vertex that must map to a must also map to a. If C is empty, then all vertices that have a directed path from a vertex that must map to a must also map to a. If a is the source element of a list S contained in a clique, then all vertices with list S that have a directed path to a vertex that must map to a list T contained in a clique with sink element a must also map to a. If a is the sink element of a list S contained in a clique, then all vertices with list S that have a directed path from a vertex that must map to a list T contained in a clique wtih source element a must also map to a. We may thus identify all these vertices that must map to a into a single vertex v that must map to a.

We claim that if this instance G with a single vertex v that must map to a has a homomorphism f to H, then this instance also has a homomorphism f' to H that maps no vertex other than v to a. If B is empty, then whenever $f(w) = a$ we may set $f'(w) = c$; if C is empty, then whenever $f(w) = a$ we may set $f'(w) = b$; if neither B nor C is empty, then whenever $f(w) = a$ we may set $f'(w) = c$ if there is a directed path from a vertex with list S contained in a clique having a as its source to w, and $f'(w) = b$ otherwise. Here we have used the fact that if list S is a clique containing a, then if S contains some vertex reachable from a then S also contains c, and if S contains some vertex that can reach a then S also contains b.

We may then replace the instance G with the instance G' obtained by removing v and updating accordingly the lists of neighbors of v. The lists will still have the desired properties since T,T' are forbidden as induced subgraphs. □

Theorem 6.3 *Let H be a reflexive antisymmetric digraph such that $U(H)$ is a split graph. If H is $C_3 - free$ then $SubRET(H)$ is polynomial time solvable. Otherwise, $SubRET(H)$ is NP-complete.*

Proof. The NP-completeness was shown before. For polynomiality, we proceed as in the preceding theorem. A split graph has vertices consisting of an independent set I and a clique K. We may as in the preceding theorem eliminate the vertices $a \in I$ from H one by one. We are left with a problem on the clique K, this time with arbitrary lists. We may greedily map source vertices of the instance to the least element of their lists. □

We have classified the subretraction problems $SubRET(H)$ for large classes of reflexive digraphs H (including symmetric digraphs [3], trees, and partial orders). Perhaps it is possible to completely classify the complexity of this problem.

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