

List Homomorphisms to Reflexive Digraphs

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Abstract

We study list homomorphism problems $L\text{-HOM}(H)$ for the class of reflexive digraphs H (digraphs in which each vertex has a loop). These problems have been intensively studied in the case of undirected graphs H , and appear to be more difficult for digraphs. However, it is known that each problem $L\text{-HOM}(H)$ is NP-complete or polynomial time solvable. In this paper we focus on reflexive digraphs. We introduce a new class of ‘adjusted interval digraphs’, point out that the list homomorphism problem $L\text{-HOM}(H)$ is polynomial time solvable when H is an adjusted interval digraph, and conjecture it is NP-complete otherwise. It suffices to prove the conjecture for digraphs H for which the underlying graph is an interval graph, and we prove the conjecture when the underlying graph of H is a clique or a tree, thus establishing a possible basis for the general result. The class of adjusted interval digraphs appears interesting in its own right, and we provide a forbidden substructure characterization which implies a polynomial time recognition algorithm.

1 Introduction

A digraph H is *reflexive* if the adjacency relation $E(H)$ is reflexive, i.e., if each vertex has a loop; it is *symmetric* if the relation $E(H)$ is symmetric.

Each digraph H is associated with two related undirected graphs. We denote by $U(H)$ the *underlying graph* of H , which has an edge uv whenever $uv \in E(H)$ or $vu \in E(H)$, and by $S(H)$ the *symmetric graph* of H , which has an edge uv whenever $uv \in E(H)$ and $vu \in E(H)$. We shall say that u is a *neighbour* of v in H , and that u, v are *adjacent* in H , if uv is an edge of $U(H)$. If u, v are adjacent in H , the pair uv is a *forward edge* if $uv \in E(H)$, a *backward edge* if $vu \in E(H)$, and a *digon* (or *symmetric edge*) if both $uv, vu \in E(H)$. Note that a loop is both a forward edge and a backward edge. An edge which is not symmetric will also be called a *single edge*. If uv is a forward edge of H we also say that u *dominates* v in H .

An *interval graph* is a graph H which admits an *interval representation*, i.e., a family of intervals $I_v, v \in V(H)$, such that $uv \in E(H)$ if and only if I_u and I_v intersect. An *interval digraph* is a digraph H which admits an *interval pair representation*, which is a family of pairs of intervals $I_u, J_u, u \in V(H)$, such that $uv \in E(H)$ if and only if I_u intersects J_v .

Note that an interval graph must be reflexive, but an interval digraph may lack loops. If the intervals $I_v, J_v, v \in V(H)$, can be chosen so that for each v the intervals I_v and J_v have the same left endpoint, we say that H is an *adjusted interval digraph*. It is again clear that an adjusted interval digraph must be reflexive.

In [2] we have studied the special case of adjusted interval digraphs H representable by intervals $I_v, J_v, v \in V(H)$, in which each interval J_v is just one point. These are called *chronological interval digraphs* [2], and we have shown that they can be characterized by the absence of certain special forbidden structures. A related class of *interval catch digraphs* has been characterized by the absence of certain other forbidden structures in [20]. One result of this paper is a forbidden structure characterization of adjusted interval digraphs.

A *homomorphism* f of a digraph G to a digraph H is a mapping $f : V(G) \rightarrow V(H)$ in which $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$ [16]. If $L(v), v \in V(G)$, are *lists* (subsets of $V(H)$), then a *list homomorphism* of G to H (with respect to the lists L) is a homomorphism satisfying $f(v) \in L(v)$ for all $v \in V(G)$. The *list homomorphism problem* $L - HOM(H)$ asks whether or not an input digraph G equipped with lists L admits a list homomorphism $f : G \rightarrow H$ with respect to L . The complexity of the list homomorphism problem $L-HOM(H)$ for undirected graphs H has been classified in [4, 5, 6] - it is polynomial time solvable when H is a so-called *bi-arc graph*, and is NP-complete otherwise. Of particular interest for this paper is the classification in the special case of reflexive graphs.

Theorem 1.1 [4] *Let H be a reflexive graph. The problem $L-HOM(H)$ is polynomial time solvable if H is an interval graph, and is NP-complete otherwise.*

This typifies the situation for graphs - the problems $L-HOM(H)$ tend to be tractable for well structured and natural classes of graphs, and NP-complete otherwise. The complexity of $L - HOM(H)$ for any digraph (or more general relational system) has been classified in [1]. The classification is complicated, (Theorem 3.1) but it does yield an algorithm to decide for any fixed digraph H whether $L - HOM(H)$ is polynomial time solvable or NP-complete. For reflexive digraphs, we have proposed a simpler graph theoretic characterization of the tractable problems [13, 8], which we verify in this paper for trees and for semi-complete digraphs. Since these special classes are the building blocks of all interval graphs [11], we believe these are important special cases to consider. Moreover, we believe that the techniques of this paper can be adapted to prove the general conjecture.

If our conjecture is true, it represents a significant simplification of Theorem 1.1 in the special case of reflexive digraphs. The tractable cases of $L - HOM(H)$ would again correspond to nicely structured digraphs H , namely the adjusted interval digraphs, which have a natural forbidden structure characterization and can be recognized in polynomial time. Moreover, they would also correspond to just one simple polymorphism of digraphs (see below). We also had a similar conjecture for the special case of irreflexive digraphs

[8, 13]. However, that conjecture has turned out to be false [14], and we shall discuss the case of irreflexive digraphs in a companion paper [14].

2 Some intractable cases

We first collect some available information about known intractable cases of $L\text{-HOM}(H)$ for reflexive digraphs H .

Theorem 2.1 *Let H be a reflexive digraph. If*

- *H contains the directed three-cycle \vec{C}_3 , or*
- *$U(H)$ contains a chordless cycle of length greater than three, or*
- *$S(H)$ is not an interval graph,*

then the problem $L\text{-HOM}(H)$ is NP-complete.

Proof. The problem $L\text{-HOM}(\vec{C}_3)$ is shown NP-complete in [8]. (All other problems $L\text{-HOM}(H)$ for reflexive digraphs with up to three vertices are known to be polynomial time solvable [8].) The NP-completeness of $L\text{-HOM}(\vec{C}_3)$ also follows from Theorem 5.1 and the remark following it.

If $U(H)$ contains a chordless cycle of length greater than three, then even a very special list homomorphism problem (the so-called "retraction problem" $\text{RET}(H)$) is NP-complete, see [7, 18]. This implies that the more general problem $L\text{-HOM}(H)$ is also NP-complete.

If $S(H)$ is not an interval graph then the undirected graph problem $L\text{-HOM}(S(H))$ is NP-complete by Theorem 1.1. Since an undirected instance G of $L\text{-HOM}(S(H))$ can be viewed as a directed graph with each edge symmetric, this implies that $L\text{-HOM}(H)$ is also NP-complete. \square

To complete the picture, we shall now show that we may restrict our attention to digraphs H for which both $S(H)$ and $U(H)$ are interval graphs.

Theorem 2.2 *If $U(H)$ is not an interval graph, then the problem $L\text{-HOM}(H)$ is NP-complete.*

Proof. If $U(H)$ is not an interval graph, then by the theorem of Lekkerkerker and Boland [11], $U(H)$ must contain a chordless cycle of length greater than three, or an asteroidal triple. In view of the last theorem, we may assume that $U(H)$ contains an asteroidal

triple, i.e., a triple of vertices $0, 1, 2$ and paths $P(0, 1), P(0, 2), P(1, 2)$ (where $P(i, j)$ joins vertices i and j), such that each vertex i from $0, 1, 2$ has no neighbours on the path joining the other two vertices.

We first recall gadgets called choosers from [4, 17], as discussed in [16]. We state the definition in a slightly more general form, and apply it to digraphs. Let i, j be distinct vertices from $0, 1, 2$ and let I, J be subsets of $\{0, 1, 2\}$. A *chooser* $Ch(i, I; j, J)$ is a digraph P with specified vertices a and b , and with lists $L(p) \subseteq V(H)$, for $p \in V(P)$, such that any list homomorphism f of P to H has $f(a) = i$ and $f(b) \in I$ or $f(a) = j$ and $f(b) \in J$; and for any $i' \in I$ and $j' \in J$ there is a list homomorphism f of P to H with $f(a) = i$ and $f(b) = i'$ and a list homomorphism g of P to H with $g(a) = j$ and $g(b) = j'$.

It is shown in [4], as explained in [16] page 174-5, that if there exist choosers $Ch(i, \{i, k\}; j, \{j, k\})$ and $Ch(i, \{i\}; j, \{k\})$, for any permutation ijk of $0, 1, 2$, then $L - HOM(H)$ is NP-complete. (Those proofs are stated in terms of undirected graphs H and choosers Ch that are paths, but they apply verbatim to arbitrary digraph choosers Ch as defined here.)

These choosers will be constructed from simpler building blocks which we call separators. A *separator* $G(i), i = 0, 1, 2$, is a digraph with two specified vertices u, v and lists $L(t), t \in V(G(i))$, such that

- every list homomorphism of $G(i)$ to H with respect to the lists L maps both u, v to i or maps neither of u, v to i , and
- for any pair of values x, y from $0, 1, 2$ in which neither or both values x, y are equal to i , there is a list homomorphism of $G(i)$ with respect to the lists L , mapping u to x and v to y .

The proof will be completed by the following two lemmas. □

Lemma 2.3 *If there exists a separator $G(i)$ for each $i = 0, 1, 2$, then the problem $L - HOM(H)$ is NP-complete.*

Proof. The separators can be used to construct the choosers as follows: $Ch(i, \{i\}; j, \{k\})$ is formed from $G(i)$ by setting $a = u$ and modifying its list to $L(u) = \{i, j\}$, and by setting $b = v$ and modifying its list to $L(v) = \{i, k\}$. To form $Ch(i, \{i, k\}; j, \{j, k\})$, we take four vertices a, b, c, d , and place one copy of $G(i)$ between a and c (identifying a with u and c with v), and another copy of $G(i)$ between b and d (identifying in a similar manner), as well as a copy of $G(j)$ between c and b and another copy of $G(j)$ between d and a . It is easy to check that the resulting digraph satisfies the conditions for a chooser $Ch(i, \{i, k\}; j, \{j, k\})$ with the specified vertices a and b . □

Lemma 2.4 *If the $U(H)$ has an asteroidal triple, then H has separators $G(i), i = 0, 1, 2$.*

Proof. Suppose H has n vertices, and $U(H)$ has an asteroidal triple $0, 1, 2$. The separator $G(i)$ will be an oriented path of length $2n$, with alternating forward and backward edges. The lists of the two end vertices of the path $G(i)$ are $\{0, 1, 2\}$. All other vertices of $G(i)$ have lists consisting of i , together with all the vertices on the path $P(j, k)$ in H (from the definition of an asteroidal triple). Note that the length of the path $G(i)$ and the orientation of its edges ensure that it admits a homomorphism (without considering the lists) that maps u and v to any two vertices of H (recall that every vertex has a loop). It follows from the definition of an asteroidal triple that any list homomorphism of $G(i)$ to H maps both u and v to i , or neither of u, v to i ; and moreover, that there are list homomorphisms of $G(i)$ to H mapping both u and v to i and both to j, k in any prescribed combination, i.e., that $G(i)$ is a separator. \square

3 Polymorphisms

Let k be a positive integer. The k -th power of H is the digraph H^k with vertex set $V(H)^k$ in which $(u_1, u_2, \dots, u_k)(v_1, v_2, \dots, v_k)$ is an edge just if each $u_i v_i$ is an edge of H . A *polymorphism of order k* is a homomorphism of H^k to H . A polymorphism f is *conservative* if $f(u_1, u_2, \dots, u_k)$ always is one of u_1, u_2, \dots, u_k . From now on we shall use the word polymorphism to mean a conservative polymorphism. A polymorphism f of order two is *commutative* if $f(u, v) = f(v, u)$ for any u, v . If H admits an ordering $<$ for which $f(u, v) = \min(u, v)$ is a polymorphism, then this polymorphism is clearly commutative; such an ordering is called a *min-ordering*. In other words, $<$ is a min-ordering of H just if it satisfies the following property: if $uv \in E(H)$ and $u'v' \in E(H)$, then $\min(u, u') \min(v, v') \in E(H)$. (A min ordering is also called an *X-underbar enumeration* [12, 16]). It is known [10] that if H admits any binary commutative polymorphism (e.g. a min-ordering), then the problem $L-HOM(H)$ is polynomial time solvable. A binary polymorphism $f : H^2 \rightarrow H$ is *commutative over the pair a, b* , if $f(a, b) = f(b, a)$. Two ternary polymorphisms also play a role in the problems $L-HOM(H)$ [1]. A polymorphism $f : H^3 \rightarrow H$ is called a *majority polymorphism* if $f(u, u, v) = f(u, v, u) = f(v, u, u) = u$ for any u, v . A ternary polymorphism $f : H^3 \rightarrow H$ is called a *Maltsev polymorphism* if $f(u, u, v) = f(v, u, u) = v$ for any u, v . A ternary polymorphism $f : H^3 \rightarrow H$ is *majority (respectively Maltsev) over a, b* , if $f(a, a, b) = f(a, b, a) = f(b, a, a) = a, f(b, b, a) = f(b, a, b) = f(a, b, b) = b$ (respectively if $f(a, a, b) = f(b, a, a) = b, f(a, b, b) = f(b, b, a) = a$).

At this point, we can state the classification of $L-HOM(H)$ due to Bulatov. The theorem applies to any relational structure H , but for our purposes we only need to state it for reflexive digraphs. Also, we formulate the result in a language of binary commutative polymorphisms in place of the more usual semi-lattice operations, since it is equivalent and is more convenient in our context.

Theorem 3.1 [1] *Let H be a reflexive digraph.*

If for every pair of vertices a, b of H there exists a polymorphism of H which is either ternary and is majority, or Maltsev, over a, b , or is binary and is commutative over a, b , then $L\text{-HOM}(H)$ is polynomial time solvable.

Otherwise, $L\text{-HOM}(H)$ is NP-complete.

We now identify an obstruction which prevents u, v from admitting a binary polymorphism which is commutative over u, v .

We define two walks $P = x_0, x_1, \dots, x_n$ and $Q = y_0, y_1, \dots, y_n$ in H to be *congruent*, if they follow the same pattern of forward and backward edges, i.e., if $x_i x_{i+1}$ is a forward (backward) edge if and only if $y_i y_{i+1}$ is a forward (backward) edge, respectively. If P and Q as above are congruent walks, we say that P *avoids* Q , if there is no edge $x_i y_{i+1}$ in the same direction (forward or backward) as $x_i x_{i+1}$.

An *invertible pair* in H is a pair of vertices u, v such that

1. there exist congruent walks P from u to v and Q from v to u , such that P avoids Q ,
2. and there exist congruent walks P' from v to u and Q' from u to v , such that P' avoids Q' .

Note that it is possible that P' is the reversal of P and Q' is the reversal of Q , as long as both P avoids Q and Q avoids P .

It will turn out to be useful to reformulate these definitions in terms of an auxiliary digraph. The *pair-digraph* H^+ associated with H has vertices $V(H^+) = \{(u, v) : u \neq v\}$, and edges $(u, v)(u', v')$, where

$$uu', vv' \in E(H) \text{ and } uv' \notin E(H), \text{ or}$$

$$u'u, v'v \in E(H) \text{ and } v'u \notin E(H).$$

Note that each vertex (u, v) of H^+ is a vertex of the square H^2 (with $u \neq v$), and each edge $(u, v)(u', v')$ of H^+ is either an edge of H^2 (if $uu', vv' \in E(H), uv' \notin E(H)$) or the reversal of an edge of H^2 (if $u'u, v'v \in E(H), v'u \notin E(H)$.) In particular, it is important to note that in the pair graph H^+ we have (u, v) dominate (u', v') in some cases where u' dominates u and v' dominates v . Now we observe that an invertible pair u, v in H yields a directed closed walk in H^+ which contains both (u, v) and (v, u) .

Lemma 3.2 *If H has an invertible pair (u, v) , then (u, v) and (v, u) belong to the same strong component C of H^+ ; moreover, for any (x, y) in C the reversed pair (y, x) also belongs to C , i.e., each pair in C is invertible.*

If H has no invertible pair, then for each strong component C of H^+ there exists a reversed strong component C' such that $(x, y) \in C$ if and only if $(y, x) \in C'$.

Proof. These properties follow from the definition of a strong component and the observation that $(u, v)(u'v') \in E(H^+)$ implies $(v', u')(v, u) \in E(H^+)$. For instance, if $(u, v), (v, u), (x, y) \in C$, then the directed closed walk containing $(u, v), (x, y)$ yields by reversal a directed closed walk containing $(v, u), (y, x)$, and by concatenation with the directed closed walk containing $(u, v), (v, u)$, we obtain a directed closed walk containing $(x, y), (y, x)$. \square

An invertible pair is an obstruction to the existence of a min ordering, in the following strong sense.

Lemma 3.3 *If u, v is an invertible pair in H , then no binary polymorphism of H is commutative over u, v .*

Proof. Suppose $(u, v)(u', v')$ is an edge of H^+ . Suppose f is a binary polymorphism of H which has $f(u, v) = f(v, u) = u$. Then we must also have $f(u', v') = u'$, and $f(v', u') = u'$. Following the directed closed walk containing (u, v) and (v, u) , we obtain a contradiction. \square

Moreover, we have the following general observation.

Theorem 3.4 *A reflexive digraph H admits a binary commutative polymorphism if and only if it has no invertible pair.*

Proof. Let H be a digraph. For each pair of distinct vertices u, v of H , we introduce a boolean variable $x_{u,v}$ that will encode the chosen value of $f(u, v)$ (say, $x_{u,v} = 0$ will mean $f(u, v) = u$ and $x_{u,v} = 1$ will mean $f(u, v) = v$). The constraints requiring f to be a polymorphism can be expressed using clauses of size two, and the commutativity means that $x_{u,v}$ is the negation of $x_{v,u}$. Thus the existence of a conservative commutative polymorphism of H can be expressed as an instance of 2-SAT, and hence decided in polynomial time (in the size of H). Moreover [19], if the 2-SAT instance has no solution then there is a cycle C of implications $(x = 1 \implies y = 1)$ containing both a variable $x_{u,v}$ and its negation $x_{v,u}$. It is easy to see from these definitions that C corresponds to a cycle in H^+ containing both (u, v) and (v, u) , i.e., that u, v is an invertible pair. \square

Finally, we relate min orderings to adjusted interval digraphs.

Theorem 3.5 *A reflexive digraph is an adjusted interval digraph if and only if it admits a min ordering.*

Proof. Given a min ordering, we can arrange the common starting points of I_v, J_v in the same order as the vertices v appear in the min ordering, and define intervals I_v and J_v as follows. If v has no forward edges towards later vertices, we end the interval I_v at the last vertex w such that vw is a double edge, and end the interval J_v at the last vertex w such that vw is a backward edge. If v has no backward edges towards later vertices, we end the interval J_v at the last vertex w such that vw is a double edge, and end the interval I_v at the last vertex w such that vw is a forward edge. (In other words, I_v ends after the last out-neighbour of v , and J_v after the last in-neighbour of v ; either the out-neighbours or the in-neighbours must be along double edges, cf. Corollary 6.2.)

Conversely, given an adjusted interval pair representation $I_v, J_v, v \in V(H)$ we obtain a min ordering of H according to the left to right order of the common left endpoints of the intervals. \square

4 Adjusted Interval Digraphs

We now strengthen Theorem 3.4 as follows.

Theorem 4.1 *A reflexive digraph H admits a min ordering if and only if it has no invertible pair.*

In fact, we shall prove the following stronger result.

Theorem 4.2 *The following statements are equivalent for a reflexive digraph H :*

1. H is an adjusted interval digraph
2. H has a min ordering
3. H has no invertible pairs
4. The vertices of H^+ can be partitioned into sets D, D' such that
 - $(x, y) \in D$ if and only if $(y, x) \in D'$
 - $(x, y) \in D$ and (x, y) dominates (x', y') in H^+ implies $(x', y') \in D$
 - $(x, y), (y, z) \in D$ implies $(x, z) \in D$.

Proof. The equivalence of 1 and 2 is proved in Theorem 3.5. Furthermore, Lemma 3.3 shows that 2 implies 3. In fact, we observe in this context that $(a, b)(a', b') \in E(H^+)$, implies that if a is ordered before b then a' must be ordered before b' . Thus a closed

directed walk in H^+ which involves both (a, b) and (b, a) means there is no min ordering. More precisely, from the definition of a min ordering, a linear ordering $<$ of $V(H)$ is a min ordering if and only if any $(a, b)(a', b') \in E(H^+)$ with $a < b$ also has $a' < b'$. It is also quite straightforward to see that 4 implies 2. It suffices to define $a < b$ if $(x, y) \in D$. Thus it remains to show that 3 implies 4.

Therefore, we assume that H has no invertible pair. Note that we may assume that H is weakly connected, otherwise we can order each weak component separately. We also note that for each strong component C of H^+ , there is a corresponding reversed strong component C' whose pairs are precisely the reversed pairs of the pairs in C ; we shall say that C, C' are *coupled* strong components. We also say that a strong component C_1 is a *predecessor* of a strong component C_2 , if there is a directed path from a vertex in C_1 to a vertex in C_2 .

The partition of $V(H^+)$ into D, D' will correspond to separating each pair of coupled strong components C, C' of H^+ . The vertices of one strong components will be placed in the set D , their reversed pairs will go to D' . We wish to make these choices in such a way as to avoid creating a *circular chain* in D , i.e., a sequence of pairs $(x_0, x_1), (x_1, x_2), \dots, (x_n, x_0) \in D$. If no circular chain is created during the construction of D , then the final D has no circular chain and the corresponding relation $<$ is transitive, i.e., yields an ordering of H .

We shall proceed as follows. Initially the sets D and D' are empty. We say that a strong component C of H^+ is *ripe* when it has no edge to another strong component in $H^+ - D$. In the general step, we shall take a ripe component C and place it in D , and simultaneously place C' and all its predecessor strong components in D' . (Note that C' need not be ripe.) We will show that there is always at least one ripe strong component which can be added to D without creating a circular chain.

The sets D, D' will always have the following properties (which are true initially). The set D has no circular chain, each strong component of H^+ belongs entirely to D, D' , or to $V(H^+) - D - D'$, the pairs in D' are precisely the reversed pairs of the pairs in D , and there is no edge of H^+ leading out of D . At the end of the algorithm each pair (x, y) with $x \neq y$ will belong either to D or to D' .

We now prove that the algorithm maintains these properties. This will mean that at the end the set D yields a min ordering.

Suppose, for a contradiction, that the addition of C creates a circular chain in D . Suppose $(x_0, x_1), (x_1, x_2), \dots, (x_n, x_0)$ is a circular chain that has occurred for the first time during the execution of the algorithm, and also suppose that at that time no shorter circular chain has occurred. Since there are no invertible pairs, we have $n \geq 3$.

Case 1. Assume that in H , there is at least one edge between the vertices x_0, x_1, \dots, x_n , say an edge $x_a x_b$. We claim that this implies that H is complete on x_0, x_1, \dots, x_n . We make the following elementary observations, assuming $j \neq i$.

1. If x_j dominates x_i then x_{j-1} dominates x_i in H .
2. If x_j dominates x_i then x_j dominates x_{i-1} in H .

To prove the first observation, we note that if x_j dominates x_i but x_{j-1} not dominate x_i in H , then (x_{j-1}, x_j) dominates (x_{j-1}, x_i) in H^+ . Since (x_{j-1}, x_j) is in $C \cup D$, the pair (x_{j-1}, x_i) must belong to $C \cup D$, implying a shorter circular chain in $C \cup D$.

To prove the second observation, we similarly note that if x_j dominates x_i but x_j does not dominate x_{i-1} in H , then (x_{i-1}, x_i) dominates (x_{i-1}, x_j) in H^+ , also implying a shorter circular chain.

Consider now the fact that x_a dominates x_b in H . Property 1 implies that $x_{a-1}, x_{a-2}, \dots, x_{b+1}$ all dominate x_b . Since x_{b+1} dominates x_b , property 2 implies that x_{b+1} dominates $x_{b-1}, x_{b-2}, \dots, x_{b+2}$, i.e., dominates all other vertices. At this point we use 1 again to derive that x_b dominates x_{b-1} , and repeated application of 2 as before implies that x_b dominates all other vertices. Continuing this way, we see that each x_j dominates all other vertices, i.e., the vertices x_0, x_1, \dots, x_n induce a complete graph in H .

We conclude the proof of Case 1 by showing that C is a trivial component (with a single vertex). If C has more than one vertex, then so does its corresponding coupled component C' , which contains the vertex (x_0, x_n) . Hence we assume for contradiction that (x_0, x_n) dominates some (a, b) not in $C \cup D$.

Up to symmetry, we may assume that x_0 dominates a in H , x_n dominates b in H and x_0 does not dominate b in H . Since (a, b) is not in $C \cup D$, the pair (x_0, x_1) , which is in C , cannot dominate (a, b) , which implies that x_1 does not dominate b in H . If x_2 dominates b in H , then (x_1, x_2) dominates (x_0, b) which dominates (a, b) in H^+ ; this is impossible, as this is a directed path starting in C and ending outside of $C \cup D$, so some edge would exit from $C \cup D$ against the rules we maintain. Therefore x_2 does not dominate b in H ; if x_3 dominates b in H , then (x_2, x_3) dominates (x_1, b) which dominates (x_0, b) which dominates (a, b) , yielding the same contradiction. Therefore x_3 does not dominate b in H , and continuing this way we would derive that x_n does not dominate b , which is false.

Thus we have $C = \{(x_n, x_0)\}, C' = \{(x_0, x_n)\}$. The same proof also shows that C' is ripe, as no (a, b) dominated by (x_0, x_n) can exist outside of $C \cup D$. It is now easy to see that if both (x_n, x_0) and (x_0, x_n) complete a circular chain with D , then D already had a circular chain.

Case 2. Assume that vertices x_0, x_1, \dots, x_n are independent in H .

Lemma 4.3 *Suppose $(x_0, x_1), (x_1, x_2), \dots, (x_n, x_0)$ is the first circular chain created by the algorithm.*

Suppose p is a vertex of H which dominates x_{i+1} but not x_i (or which is dominated by x_{i+1} but not by x_i).

Then $(x_0, x_1), \dots, (x_i, p), (p, x_{i+2}), \dots, (x_n, x_0)$ is also a circular chain created at the same time.

Proof. Suppose p is a vertex of H which dominates x_{i+1} but not x_i , or which is dominated by x_{i+1} but not by x_i . Then (x_i, x_{i+1}) dominates (x_i, p) in H^+ , and since (x_i, x_{i+1}) is in $C \cup D$, we must also have (x_i, p) in $C \cup D$. Furthermore, since x_{i+1} does not dominate or is dominated by x_{i+2} in H , we also have (x_{i+1}, x_{i+2}) dominating (p, x_{i+2}) , whence (p, x_{i+2}) is in $C \cup D$. In conclusion, we see that any such vertex p can replace x_{i+1} in the circular chain $(x_0, x_1), (x_1, x_2), \dots, (x_n, x_0)$. \square

This will be useful if we know that p dominates or is dominated by one of $x_0, x_1, \dots, x_i, x_{i+2}, \dots, x_n$, because in this situation we would have a circular chain on a set of vertices which is not independent in H , and hence we would obtain a contradiction by Case 1. One such use occurs in the following lemma.

Lemma 4.4 *Suppose $(x_0, x_1), (x_1, x_2), \dots, (x_n, x_0)$ is the first circular chain created by the algorithm.*

- *If p is a vertex of H which dominates x_j and x_k with $j \neq k$. Then p dominates each x_i .*
- *If p is a vertex of H which is dominated by x_j and x_k with $j \neq k$. Then p is dominated by each x_i .*
- *If p is a vertex of H which dominates x_j and is dominated by x_k with $j \neq k$. Then p both dominates and is dominated by each $x_i, i \neq j, k$.*

Proof. If p dominates x_{i+1} but not x_i , then Lemma 4.3 implies that p can replace x_{i+1} in the circular chain; however at least one of x_j, x_k is not equal to x_{i+1} , whence the vertices of the chain are not independent and we conclude by Case 1. The other items are proved similarly. \square

We now claim that the circular chain $(x_0, x_1), (x_1, x_2), \dots, (x_n, x_0)$ has at most one pair, say (x_n, x_0) , in C (with all other pairs in D). Otherwise, assume some (x_i, x_{i+1}) is also in the strong component C , and let P be a directed path in C from (x_n, x_0) to (x_i, x_{i+1}) . Let the penultimate pair on this path be (p, q) , and, without loss of generality, assume that $px_i, qx_{i+1} \in E(H), px_{i+1} \notin E(H)$. (In the case $x_i p, x_{i+1} q \in E(H), x_{i+1} p \notin E(H)$, the argument is symmetric.) We first claim that p does not dominate any x_j with $j \neq i$. Indeed, p does not dominate x_{i+1} since (p, q) dominates (x_i, x_{i+1}) in H^+ . If p dominates x_{i+2} , then Lemma 4.3 allows replacing x_{i+2} by p , which reduces this to Case 1 since p dominates x_i . Similarly, p does not dominate x_{i+3} and so on. Next we claim that q does

not dominate x_i . Indeed, if q dominates x_i , then Lemma 4.4 implies that q dominates all x_j (since q and x_{i+1} are not independent. This is a contradiction, since it would mean that (p, q) dominates (x_i, x_{i+2}) in H^+ , implying that (x_i, x_{i+2}) is in $C \cup D$ and thus there is a shorter circular chain in H . Therefore q does not dominate x_i . By a double application of Lemma 4.3, we conclude that we can replace x_i and x_{i+1} by p and q in the circular chain in H . Continuing this way, we replace (p, q) by the previous pair on the path P , until we obtain the pair (p', q') which is the first pair after (x_n, x_0) . Since x_0 is adjacent to q' , we are back in Case 1.

Thus the circular chain $(x_0, x_1), (x_1, x_2), \dots, (x_n, x_0)$ has only the pair (x_n, x_0) in C , and any circular chain in $C \cup D$ has exactly one pair in C .

The digraph H turns out to have a very special structure. We claim that in this situation there exists a non-empty set K of vertices of H such that $H \setminus K$ has weak components C_1, C_2, \dots, C_m , where $x_i \in C_i, i = 1, 2, \dots, n$, and such that if $p \in K$ dominates (respectively is dominated by) a vertex in C_i , then p dominates (respectively is dominated by) all vertices in C_i ; moreover, if x'_0, x'_1, \dots, x'_n are any vertices with $x'_i \in C_i$, then $(x'_0, x'_1), (x'_1, x'_2), \dots, (x'_n, x'_0)$ is also a circular chain.

Indeed, we let K consist of all vertices of H that dominate each x_i , or are dominated by each x_i . It is easy to see that K must be non-empty, as Lemma 4.4 implies that any p dominated by $x_j, x_k, j \neq k$ belongs to K . If such a p did not exist, we could replace x_j by its neighbour p on a path joining x_j to x_k by Lemma 4.3. The new independent set $x_0, \dots, x_{j-1}, p, x_{j+1}, \dots, x_n$ must also have empty K , by Lemma 4.4 and the fact that $n \geq 3$. Continuing this way, we obtain a new independent set in which some x_j and x_k do have a common neighbour, and hence a contradiction.

The same argument shows that two different x_j, x_k cannot lie in the same weak component C_i of $H \setminus K$, as any path joining x_j to x_k was shown to contain a vertex of K . Therefore we can number the components so that C_i contains x_i for $i = 1, 2, \dots, n$. (There may be additional components C_i with $i = n + 1, \dots, m$.) Now Lemma 4.3 implies that each x_i can be replaced by any neighbour in C_i , thus any vertex of C_i can be taken as x_i . Thus each $p \in K$ that dominates a vertex in C_i also dominates all vertices in C_i , and similarly for vertices p dominated by a vertex in C_i .

This creates a situation where any pair (y, y') in the strong component C of H^+ containing (x_n, x_0) must satisfy $y \in C_n, y' \in C_0$. This easily implies that the strong component C does not have any arcs entering it from the outside, and hence the strong component C' coupled with C is also ripe. We claim that C' cannot complete a circular chain with D . Otherwise, the pair (x_0, x_n) would also complete a circular chain by the same argument. Thus both (x_0, x_n) and (x_n, x_0) complete a circular chain with D , whence D must already contain a circular chain, a contradiction.

Of course, if the addition of C' does not create a circular chain, then we add C' to D and C to D' . □

This gives us a desired polynomially checkable forbidden subgraph characterization of adjusted interval digraphs. As noted above, checking for invertible pairs amounts to computing the strong components of H^+ and checking for the existence of a pair $(u, v), (v, u)$ in one strong component.

Corollary 4.5 *Let H be a reflexive digraph. Then H is an adjusted interval digraph if and only if it has no invertible pair.* \diamond

5 Semi-complete Digraphs with Invertible Pairs

Theorem 5.1 *Suppose H is a reflexive semi-complete digraph. If H contains an invertible pair, then $L\text{-HOM}(H)$ is NP-complete.*

Proof. We will appeal to Bulatov's characterization, Theorem 3.1, showing that if there exist invertible pairs in H , then some invertible pair a, b admits no polymorphism as prescribed by Theorem 3.1.

It turns out that some structures in H limit our choices of polymorphisms from the theorem. Let R be the reflexive digraph $V(R) = \{a, b, c\}$ and $E(R) = \{aa, bb, cc, ab, bc, ac, ca\}$.

Lemma 5.2 *There is no polymorphism g on the digraph R which is a majority over a, b .*

Proof. Suppose g is a polymorphism of R which is a majority over a, b , i.e., $g(a, a, b) = g(a, b, a) = g(b, a, a) = a$, and $g(a, b, b) = g(b, a, b) = g(b, b, a) = b$. We claim that g must also be a majority over b, c . Note that $g(c, c, b)g(a, a, b) = g(c, c, b)a \in E(R)$. Hence $g(c, c, b) = c$, as b does not dominate a in R . Similarly, $g(c, b, c) = g(b, c, c) = c$. Also $g(b, b, c)g(b, b, a) = g(b, b, c)b \in E(R)$ thus $g(b, b, c) = b$ and similarly $g(b, c, b) = g(c, b, b) = b$. Now we can conclude that g is also majority over a, c , using the fact that $g(a, a, c)g(b, b, c) \in E(R)$ and $g(b, b, c)g(c, c, a) \in E(R)$.

Now we note that we have $g(a, b, c)g(b, b, c) = g(a, b, c)b \in E(R)$, which implies that $g(a, b, c) \in \{a, b\}$ (since c doesn't dominate b in R); we have $g(a, b, b)g(a, b, c) \in E(R)$, which similarly implies that $g(a, b, c) \in \{b, c\}$; and we have $g(c, a, c)g(a, b, c) \in E(R)$, which similarly implies that $g(a, b, c) \in \{a, c\}$, which is impossible. \square

Lemma 5.3 *Suppose H is a reflexive digraph with $ab \in E(H), ba \notin E(H)$. There is no polymorphism h over the digraph H which is a Maltsev operation over a, b .*

Proof. If h is Maltsev over a, b , then $h(a, a, b)h(a, b, b) = ba \in E(H)$, a contradiction. \square

Thus let us assume H contains invertible pairs. If H also contains an induced reflexive directed three-cycle \vec{C}_3 , then $\text{L-HOM}(H)$ is known to be NP-complete. Thus we may assume for the proof that H does not contain \vec{C}_3 . By a similar argument, we may assume that $S(H)$ is an interval graph, and in particular, $S(H)$ does not contain an induced four-cycle.

If H contains invertible pairs, then there exist directed closed walks $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n), (x_0, y_0)$ in H^+ which contains both (a, b) and (b, a) for some $a, b \in V(H)$. We say that such a closed walk C is an *inverting cycle* for the pair a, b . As noted in Lemma 3.2, each vertex (x_i, y_i) of C is itself invertible.

An inverting cycle C in H^+ consists of forward edges only. Recall that, in H , these edges could correspond to pairs of edges $x_i x_{i+1}, y_i y_{i+1}$, which are either forward or backward.

We first assume that for some C and some i we have the edges $x_i x_{i+1}, x_{i+2} x_{i+1} \in E(H)$ and $y_i y_{i+1}, y_{i+2} y_{i+1} \in E(H)$. Without loss of generality, let us assume $i = 0$, i.e., that $x_0 x_1, x_2 x_1, y_0 y_1, y_2 y_1 \in E(H)$ and $x_0 y_1, y_2 x_1 \notin E(H)$. Therefore, the pair (x_0, y_1) dominates (x_1, y_1) and is dominated by (x_0, y_0) , which are consecutive in the cycle C . Thus we may assume that (x_0, y_1) is also in C , and hence is invertible. The same argument shows that (x_1, y_2) is also invertible.

Since H is semi-complete, we must have $y_1 x_0, x_1 y_2 \in E(H)$. If $y_1 x_1 \notin E(H)$, then y_1, x_0, x_1 are all distinct and must induce R , since there is no induced \vec{C}_3 . Then over y_1, x_0 there is no majority by Lemma 5.2, no Maltsev by Lemma 5.3, and no commutative binary polymorphism by Lemma 3.3. Hence $\text{L-HOM}(H)$ is NP-complete by Theorem 3.1.

If $y_1 x_1 \in E(H)$, then y_1, x_1, y_2 must be distinct and the same argument as above implies that $x_1 y_1 \in E(H)$. Then the same argument again applied to the triple y_1, x_0, x_1 implies that $x_1 x_0 \in E(H)$, and applied to the triple y_1, x_1, y_2 implies that $y_1 y_2 \in E(H)$. Note that $x_0 \neq y_2$ because $x_0 y_1 \notin E(H)$ but $y_2 y_1 \in E(H)$. If $y_2 x_0 \notin E(H)$ then we have a copy of R over x_0, y_1, y_2 ; if $x_0 y_2 \notin E(H)$ then we have a copy of R over x_0, x_1, y_2 . This yields an induced four-cycle x_0, x_1, y_1, y_2, x_0 in $S(H)$, contrary to our assumption.

Thus we may assume that on any inverting cycle all edges go in the same direction, forward or backward. Without loss of generality, assume that on C all edges $x_i x_{i+1}$ in H are forward (and similarly for $y_i y_{i+1}$). If there is a copy of \vec{C}_3 or R , the problem $\text{L-HOM}(H)$ is NP-complete as above. Otherwise, we claim that all $x_i y_i \in E(H)$ and $y_i x_i \in E(H)$. Indeed if $y_i x_i \notin E(H)$, then a copy of \vec{C}_3 or R exists on x_{i-1}, x_i, y_i , unless $x_i = x_{i-1}$. Note that if $x_i = x_{i-1}$ would mean that $x_i y_i \notin E(H)$ also holds, contrary to the fact that H is semi-complete. If $x_i y_i \notin E(H)$, then on some inverting cycle involving the invertible pair x_i, y_i , the same argument would show the existence of \vec{C}_3 or R . \square

We note that this result implies the first bullet of Theorem 2.1, since a reflexive directed three-cycle with edges ab, bc, ca is a semi-complete graph with an invertible pair - say ac :

indeed, ac, cb, ca, ab, ca is a directed cycle in \vec{C}_3^+ . In fact, if we succeed in proving that any reflexive graph H with an invertible pair has its list homomorphism problem $L\text{-HOM}(H)$ NP-complete, the other bullets of Theorem 2.1 will follow in a similar fashion.

Corollary 5.4 *Let H be a reflexive semi-complete graph.*

If H is an adjusted interval digraph, then the problem $L\text{-HOM}(H)$ is polynomial time solvable.

Otherwise $L\text{-HOM}(H)$ is NP-complete.

6 Trees

We believe that the methods of the previous section can be generalized to prove that any reflexive digraph H which contains an invertible pair yields an NP-complete problem $L\text{-HOM}(H)$. For the time being, we shall in this section verify our conjecture for reflexive digraphs H for which $U(H)$ is a tree. It is well known [11] that a tree is an interval graph if and only if it is a caterpillar, i.e., if the removal all leaves yields a path. Let $S(x)$ denote the set of leaves of H adjacent to the vertex $x \in P$. As usual, we refer to H as a tree, or star, etc., to mean that $U(H)$ (without the loops) is a tree, or star, etc., respectively.

We begin by stating a more convenient definition of a min ordering for reflexive digraphs.

Lemma 6.1 *Let H be a reflexive digraph. Then an ordering $<$ of $V(H)$ is a min ordering if and only if for any three vertices $i < j < k$ we have*

- $ik \in E(H)$ implies $ij \in E(H)$, and
- $ki \in E(H)$ implies $ji \in E(H)$.

Proof. The necessity of the two properties follows by taking the edge ik (respectively ki) and the loop at j . To see the sufficiency, consider edges $xy, x'y'$ of H and assume without loss of generality that $x < x', y' < y$; thus $\min(x, x') \min(y, y') = xy'$. If $x = y'$, then xy' is an edge since H is reflexive. If $x < y'$, then xy' is an edge because of the triple $x < y' < y$. If $y' < x$, then xy' is an edge because of the triple $y' < x < x'$. \square

Corollary 6.2 *Let H be a reflexive digraph. An ordering of the vertices of H is a min ordering if and only if for each vertex v the vertices that follow v in the ordering consist of*

1. first, all vertices that are adjacent to v by double edges,
2. second, all vertices that are adjacent to v by single edges, either all forward or all backward, and
3. last, all vertices that have no edges to or from v .

Of course, any of the three groups could be empty. Note that, in particular, in a min ordering of H it cannot be the case that a vertex v has both forward and backward edges towards vertices that follow it in the ordering.

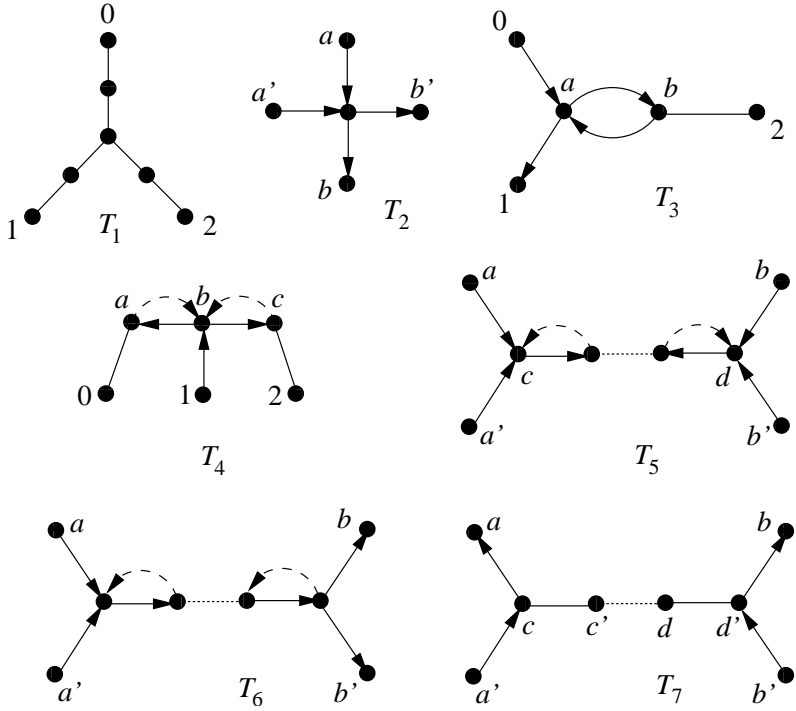
If H is a star, we shall define H to be a *good caterpillar*, if it does not contain, as induced subgraph, the tree T_2 depicted below. If H is not a star, we define it to be a *good caterpillar* if it has a longest path $v_0, v_1, \dots, v_k, v_{k+1}$ satisfying the following conditions for all i . (Note that v_1, v_2, \dots, v_k is the path P , and that $v_0 \in S(v_1), v_{k+1} \in S(v_k)$.)

1. If $v_i v_{i+1} \in E(H)$, then $v_i v \in E(H)$ for all $v \in S(v_i) - v_{i-1}$.
2. If $v_{i+1} v_i \in E(H)$, then $v v_i \in E(H)$ for all $v \in S(v_i) - v_{i-1}$.

Note that if $v_i v_{i+1}$ is a double edge then so are all $v_i v, v \in S(v_i) - v_{i-1}$. Observe that there are no restrictions on v_0 , other than those arising from the restrictions on v_1 . Indeed, all edges $v_1 v$ for $v \in S(v_1) - v_0$ must follow the direction of the edge $v_1 v_2$ (forward, backward, or double) - with the possible exception of a single vertex v , which must be the vertex v_0 . Thus such a v_0 can be chosen if and only if the restrictions on v_1 have at most one exception. Similarly, there are no restrictions on v_{k+1} , other than those arising from the restrictions on v_k . All edges $v_k v$ for $v \in S(v_k)$ must follow the direction of the edge $v_k v_{k+1}$. It is easy to see that such a v_{k+1} can be chosen if and only if between v_k and $S(v_k)$ there does not exist at the same time a single forward and a single backward edge. Finally, we note that the exceptional case, when H is a star, also conforms to the general definition; we have chosen to state it separately only for convenience.

Theorem 6.3 *Let H be a reflexive digraph that is a tree. Then the following statements are equivalent.*

1. H is a good caterpillar
2. H is an adjusted interval digraph
3. H does not contain (as an induced subgraph) any of the trees T_1, \dots, T_7 or their reverses.



Proof. The edges in the trees T_1, \dots, T_7 that are not oriented can be forward, backward, or double; the dashed edges are optional.

We shall show that 1 implies 2, 2 implies 3, and 3 implies 1. Indeed, 1 implies 2 via Theorem 3.5, as a good caterpillar can be ordered starting from v_0 and proceeding to v_1, v_2, \dots, v_k , with listing the double edges of $S(v_i) - v_{i-1}$ first, as suggested by Corollary 6.2. The definition of a good caterpillar ensures that the listing for $S(v_i) - v_{i-1}$ can be chosen to end with v_{i+1} .

Theorem 3.5 also allows us to derive 3 from 2: none of the forbidden subtrees allows a min ordering. To see this, in the trees T_1, T_3, T_4 focus on the vertices 0, 1, 2, and on the trees T_2, T_5, T_6, T_7 focus on the vertices a, a', b, b' .

It remains to show that 3 implies 1. Thus suppose H is a reflexive tree which does not contain any of $T_1 - T_7$ or their reverses. Since H does not contain T_1 it is a caterpillar. If H is a star, the conclusion now follows. Thus assume H is not a star: when all leaves of H are removed we obtain a path P , say $P = p, r, s, \dots, y, z$. We will prove that one of p, z can be chosen as v_1 and the other as v_k . Suppose first that p cannot be chosen to satisfy the condition for v_1 . Then in $S(p)$ there must be two vertices v, v' such that the edges pv, pv' do not follow the direction of the edge pr on P . If pr is a double edge, this means that pv, pv' are single edges. Since H does not contain T_3 , both are forward (or both backward) edges. This implies that all edges $pv, v \in S(p)$ follow the direction of pr ,

and thus p can be chosen to satisfy the condition for v_k . Similarly, if pr is a single (forward or backward edge), p can be chosen as v_k , since H does not contain T_2 . Therefore, each of p, z satisfies the condition for v_1 or for v_k . Suppose next that neither p nor z satisfy the condition for v_1 . Then each contains two single edges whose direction does not follow the direction of pr ; this contradicts the fact that H does not contain T_5 and T_6 or their reverses. Similarly, the absence of T_7 implies that at least one of p, z satisfies the condition for v_k . The absence of T_4 (and its reverse) implies that each intermediate vertex r, s, \dots, y of P satisfies the condition for v_i if its left or its right neighbour on P plays the role of v_{i+1} . Finally, if one vertex of P requires its left neighbour, while another requires its right neighbour, we again obtain a contradiction as above with the fact that H does not contain the trees T_5, T_6, T_7 . \square

We now recall and enhance the *indicator construction* from [15], cf. [16]. For a fixed indicator I, i, j (that is a digraph I with two specified vertices i, j) in which each vertex v has a list $L(v) \subseteq V(H)$, the indicator construction transforms a digraph H into the digraph H^* , with the same vertex set as H , and with adjacency defined by the following rule: xy is an edge of H^* just if there exists a list homomorphism of I to H that maps i to x and j to y . It is easy to see that the following extension of Lemma 5.5 of [16] holds. (The proof is identical, with the obvious addition of lists; note that we assume that the lists of the vertices i and j are the entire set $V(H^*)$: this ensures that the proof in [16] properly applies to reduce $L-HOM(H^*)$ to $L-HOM(H)$).

Lemma 6.4 *If the problem $L-HOM(H^*)$ is NP-complete, and if $L(i) = L(j) = V(H^*)$, then the problem $L-HOM(H)$ is also NP-complete.*

We now apply these tools to prove the following dichotomy.

Corollary 6.5 *Let H be a reflexive digraph that is a tree.*

If H is a good caterpillar, then $L-HOM(H)$ is polynomial time solvable. Otherwise, $L-HOM(H)$ is NP-complete.

Proof. If H is a good caterpillar, the theorem implies that it has a min ordering and hence $L-HOM(H)$ is polynomial time solvable. Otherwise, the theorem implies that H contains T_1, T_2, \dots , or T_7 .

If H contains T_1 , then $S(H)$ is not an interval graph and hence $L-HOM(H)$ is NP-complete by Theorem 2.1.

If H contains T_2 , then we shall apply Lemma 6.4. Consider the indicator I consisting of three vertices i, c, j and two edges ic, cj , with the lists $L(i) = L(j) = \{a, a', b, b'\}$, $L(c) = V(H)$. It is clear that H^* is a reflexive digraph that is a cycle with four vertices.

Thus $L - HOM(H^*)$ is NP-complete by Theorem 2.1, and $L - HOM(H)$ is NP-complete by Lemma 6.4.

If H contains T_3 then consider the three vertices $0, 1, 2$ of T_3 . We shall prove that $L - HOM(H)$ is NP-complete using Lemma 2.3. Indeed, since there is a path joining $0, 1$ that avoids the neighbours of 2 , the separator $G(2)$ is constructed as in Lemma 2.4. To construct $G(1)$, we take a path that begins with a forward and then a double edge, followed by a sufficiently long alternating sequence of forward and backward edges, and ending with a double edge followed by a backward edge. The lists will be $\{0, 1, 2\}$ everywhere except a will be added to the lists of the second and second to last vertex and b will be added to the third and third to last vertex. This pattern of edges and lists ensures that there is a list homomorphism mapping the first vertex of $G(1)$ to 0 and the last vertex to 2 and conversely, while if the first or last vertex of $G(1)$ is mapped to 1 , the entire path must map to 1 . The path $G(0)$ is constructed similarly. By Lemma 2.3, $L - HOM(H)$ is NP-complete.

If H contains T_4 , we proceed similarly, Only $G(1)$ requires an explanation: it is enough to take a sufficiently long path of alternating forward and backward edges with a middle vertex t of indegree zero, and assign the lists $\{0, 1, 2\}$ to the end vertices, the lists $\{0, 1, 2, a, c\}$ to all inner vertices except t , and the list $\{0, 1, 2, a, b, c\}$ for the special vertex t . It is again easy to check that this pattern of forward and backward edges, together with the lists, ensure the required properties for the separator $G(1)$.

If H contains T_5 , we shall again use Lemma 6.4. The indicator will be a path I from i to j identical to the path a, c, \dots, d, b in T_5 . the lists are $L(i) = L(j) = \{a, a', b, b'\}$ and otherwise $L(x) = \{x, a, a', b, b'\}$. It is easy to check that H^* is a reflexive cycle with four vertices. The proof for T_6 is similar.

Consider now the last tree T_7 . If the edge cc' or dd' is double, T_7 contains T_3 and hence we are done. Thus we shall assume that $c'c, dd'$ are forward edges. (By relabeling we obtain the case when they are both backward edges; the case when one is forward and the other backwards is different, but the proof is similar.) We again proceed to use Lemma 6.4. The indicator will be a path I from i to j consisting of a path from i to a middle vertex t identical to the path a, c, c', \dots, d', b' in T_7 , followed by a path from t to j identical to the path a', d', d, \dots, c, b in T_7 . The lists are $L(i) = L(t) = L(j) = \{a, a', b, b'\}$ and $L(x) = V(H)$ otherwise. It is easy to check that H^* is the reflexive cycle with edges $ab, ab', a'b, a'b'$. (The path from i to t ensures the presence of the edges $ab, a'b$ and the path from t to j ensures the presence of the edges $ab', a'b'$.) \square

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