# Adjusted Interval Digraphs and Complexity of List Homomorphisms

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**Abstract**

Interval digraphs were introduced by West et all. They can be recognized in polynomial time and admit a characterization in terms of incidence matrices. Nevertheless, they do not have a forbidden structure characterization nor a low-degree polynomial time recognition algorithm.

We introduce a new class of 'adjusted interval digraphs', obtained by a slight change in the definition. By contrast, these digraphs have a natural forbidden structure characterization, parallel to a characterization for undirected graphs, and admit an easy recognition algorithm.

Adjusted interval digraphs arise as natural analogues of interval graphs in the context of list homomorphism problems. Each digraph $H$ defines a corresponding list homomorphism problem LHOM($H$). For undirected graphs, it is known that reflexive interval graphs $H$ yield polynomially solvable problems LHOM($H$), while LHOM($H$) is NP-complete for all other reflexive graphs $H$. We observe that if $H$ is an adjusted interval digraph, then the problem LHOM($H$) is also polynomial time solvable, and we conjecture that for all other reflexive digraphs $H$ the problem LHOM($H$) is NP-complete. We prove the conjecture in two basic cases.

## 1 Introduction

This is an expanded version of the extended abstract [11]. In that note we announced the main result and some of its applications; the proofs were not included, except for an explanation of our strategy for the proof of Theorem 3.2. Here we include all proofs and provide additional applications.

An *interval graph* [13] is a graph $H$ which admits an *interval representation*, i.e., a family of intervals $I_v, v \in V(H)$, such that $uv \in E(H)$ if and only if $I_u$ and $I_v$ intersect. A digraph analogue has been defined in [26] - an *interval digraph* is a digraph $H$ which admits an *interval pair representation*, which is a family of pairs of intervals $I_v, J_v, v \in V(H)$, such that $uv \in E(H)$ if and only if $I_u$ intersects $J_v$. Interval graphs admit elegant characterizations [23, 12], cf. [13], and linear time recognition algorithms [1, 15, 4]. By contrast, the class of interval digraphs so far lacks comparable simple forbidden structure characterizations, and the best algorithm for their recognition to date is a dynamic programming algorithm of complexity $O(nm^6(n+m)\log n)$ [24]. Motivated by the study of list homomorphisms (as explained below), we introduce a new digraph analogue of interval graphs, and argue that it has much nicer properties than the usual interval digraphs. Indeed, we will prove a simple forbidden structure characterization (directly implying a polynomial time recognition algorithm). An *adjusted interval digraph* is an interval digraph $H$ that admits an interval pair representation $I_v, J_v, v \in V(H)$, in which the intervals $I_v$ and $J_v$ have the same left endpoint.

Note that the definition of an interval graph implies that an interval graph is *reflexive* (each vertex has a loop). Interval digraphs in the classical sense may lack loops. (If the intervals $I_v, J_v$ are disjoint there is no loop at $v$.) However, an adjusted interval digraph must again be reflexive.

In [5] we studied the special case of adjusted interval digraphs $H$ representable by intervals $I_v, J_v, v \in V(H)$, in which each interval $J_v$ is just one point. These are called *chronological interval digraphs* [5], and we have shown that they can be characterized by the absence of certain special forbidden structures. In [25], a related class of *interval catch digraphs* has been characterized by the absence of certain other forbidden structures. Here we provide a forbidden structure characterization of adjusted interval digraphs, directly implying a direct polynomial time a simple polynomial time recognition algorithm for the class of adjusted interval digraphs. The forbidden structure we introduce, an invertible pair, is also useful in a number of similar situations [17, 18]. In particular, for the class of interval graphs [18], the resulting characterization is equivalent to the well known characterizations in terms of induced cycles and asteroidal triples [23], or in terms of consecutive clique enumerations [12].

Each digraph $H$ is associated with two related undirected graphs. We denote by $U(H)$ the *underlying graph* of $H$, which has an edge $uv$ whenever $u \neq v$ and $uv \in E(H)$ or $vu \in E(H)$, and by $S(H)$ the *symmetric graph* of $H$, which has an edge $uv$ whenever $u \neq v$ and $uv \in E(H)$ and $vu \in E(H)$. Note that the loops of $H$, if any, are removed from both $U(H)$ and $S(H)$.

Adjusted interval digraphs are motivated by the study of the complexity of list homomorphisms. A *homomorphism* $f$ of a digraph $G$ to a digraph $H$ is a mapping $f : V(G) \to V(H)$ in which $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$ [20]. If $L(v), v \in V(G)$, are *lists* (subsets of $V(H)$), then a *list homomorphism* of $G$ to $H$ (with respect to the lists $L$) is a homomorphism satisfying $f(v) \in L(v)$ for all $v \in V(G)$. The *list homomorphism problem* $LHOM(H)$ asks whether or not an input digraph $G$ equipped with lists $L$ admits a list homomorphism $f : G \to H$ with respect to $L$. The complexity of the list homomorphism problem LHOM($H$) for undirected graphs $H$ has been classified in [6, 7, 8].

Of particular interest for this paper is the classification in the special case of reflexive graphs.

**Theorem 1.1** *[6] Let $H$ be a reflexive graph.*

*If $H$ is an interval graphs, then the problem LHOM($H$) is polynomial time solvable.*

*Otherwise, the problem LHOM(H) is NP-complete.*

The complexity of $LHOM(H)$ for any digraph (and more general relational structure) has been classified in [2] (see Theorem 4.1). For reflexive digraphs $H$, we propose a simpler classification. Specifically, we observe that each adjusted interval digraph $H$ has polynomial time solvable list homomorphism problem LHOM($H$), and conjecture that for any other reflexive digraph $H$ the problem LHOM($H$) is NP-complete. Thus it appears that in the context of list homomorphisms, adjusted interval digraphs $H$ play the same role for digraphs as interval graphs $H$ play for graphs - namely they exactly identify the tractable cases of LHOM($H$).

We show that it suffices to verify the conjecture for digraphs whose underlying graphs are interval graphs. Then we proceed to verify it for digraphs whose underlying graphs are complete graphs and trees, which can be viewed as the building blocks of interval graphs.

## 2 Invertible Pairs

The *underlying graph* of $H$ has an edge $uv$ whenever $uv \in E(H)$ or $vu \in E(H)$. If $u, v$ are adjacent in the underlying graph of $H$, the pair $uv$ is a *forward edge* if $uv \in E(H)$, and a *backward edge* if $vu \in E(H)$. Note that a loop is both a forward edge and a backward edge. If $uv \in E(H)$, we say that $u$ *dominates* $v$ in $H$.

We define two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in $H$ to be *congruent*, if they follow the same pattern of forward and backward edges, i.e., if $x_i x_{i+1}$ is a forward (backward) edge if and only if $y_i y_{i+1}$ is a forward (backward) edge, respectively. If $P$ and $Q$ as above are congruent walks, we say that $P$ *avoids* $Q$, if there is no edge $x_i y_{i+1}$ in the same direction (forward or backward) as $x_i x_{i+1}$.

An *invertible pair* in $H$ is a pair of vertices $u, v$ such that

- there exist congruent walks $P$ from $u$ to $v$ and $Q$ from $v$ to $u$, and such that $P$ avoids $Q$,
- there exist congruent walks $P'$ from $v$ to $u$ and $Q'$ from $u$ to $v$, such that $P'$ avoids $Q'$.

It will turn out to be useful to reformulate these definitions in terms of an auxiliary digraph. The *pair-digraph* $H^+$ associated with $H$ has vertices $V(H^+) = \{(u,v) : u \neq v\}$, and edges $(u,v)(u',v')$, where

$$uu', vv' \in E(H) \text{ and } uv' \notin E(H), \text{ or}$$

$$u'u, v'v \in E(H) \text{ and } v'u \notin E(H).$$

**Lemma 2.1** *If $H$ has an invertible pair $(u,v)$, then $(u,v)$ and $(v,u)$ belong to the same strong component $C$ of the pair-digraph $H^+$; moreover, for any $(x,y)$ in $C$ the reversed pair $(y,x)$ also belongs to $C$, i.e., each pair in $C$ is invertible.*

*If $H$ has no invertible pair, then for each strong component $C$ of $H^+$ there exists a* reversed strong component $C'$ *such that $(x,y) \in C$ if and only if $(y,x) \in C'$.*

**Proof.** These properties follow from the definition of a strong component and the observation that $(u,v)(u'v') \in E(H^+)$ implies $(v',u')(v,u) \in E(H^+)$. For instance, if $(u,v),(v,u),(x,y) \in C$, then the directed closed walk containing $(u,v),(x,y)$ yields by reversal a directed closed walk containing $(v,u),(y,x)$, and by concatenation with the directed closed walk containing $(u,v),(v,u)$, we obtain a directed closed walk containing $(x,y),(y,x)$. □

An ordering $<$ of the vertices of $H$ is a *min ordering* of $H$ if it satisfies the following property: if $uv \in E(H)$ and $u'v' \in E(H)$, then $\min(u,u')\min(v,v') \in E(H)$. (A min ordering was also called an *$X$-underbar enumeration* [14, 20]). The following result relates min orderings to adjusted interval digraphs.

**Theorem 2.2** *A reflexive digraph is an adjusted interval digraph if and only if it admits a min ordering.*

It is interesting to note that a reflexive undirected graph $H$ has a min ordering if and only if it is an interval graph [6]. Thus Theorem 2.2 also provides motivation in favour of adjusted interval digraphs.

**Proof.** Given a min ordering, we can arrange the common starting points of $I_v, J_v$ in the same order as the vertices $v$ appear in the min ordering, and define intervals $I_v$ and $J_v$ as follows. If $v$ has no forward edges towards later vertices, we end the interval $I_v$ at the last vertex $w$ such that $vw$ is a double edge, and end the interval $J_v$ at the last vertex $w$ such that $vw$ is a backward edge. If $v$ has no backward edges towards later vertices, we end the interval $J_v$ at the last vertex $w$ such that $vw$ is a double edge, and end the interval $I_v$ at the last vertex $w$ such that $vw$ is a forward edge. Conversely, given an adjusted interval pair representation $I_v, J_v, v \in V(H)$ we obtain a min ordering of $H$ according to the left to right order of the common left endpoints of the intervals. □

Min orderings also play an important role for list homomorphism problems, cf. [20].

**Theorem 2.3** *[14] If $H$ admits a min ordering, then the problem $LHOM(H)$ is polynomial time solvable.*

Finally, we observe that an invertible pair is an obstruction to the existence of a min ordering.

**Lemma 2.4** *If $H$ has an invertible pair, then $H$ does not admit a min ordering.*

**Proof.** Suppose $(u, v)(u', v')$ is an edge of the pair-digraph $H^+$. Suppose $<$ is a min ordering of $H$, and suppose $u < v$. The we must also have $u' < v'$. Following the directed closed walk in $H^+$ which contains $(u, v)$ and $(v, u)$, we obtain a contradiction. □

# 3 Adjusted Interval Digraphs

We now strengthen Lemma 2.4.

**Theorem 3.1** *A reflexive digraph $H$ admits a min ordering if and only if it has no invertible pair.*

In fact, we shall prove the following stronger result.

**Theorem 3.2** *The following statements are equivalent for a reflexive digraph $H$:*

1. *$H$ is an adjusted interval digraph*
2. *$H$ has a min ordering*
3. *$H$ has no invertible pairs*
4. *The vertices of $H^+$ can be partitioned into sets $D, D'$ such that*
   - *$(x, y) \in D$ if and only if $(y, x) \in D'$*
   - *$(x, y) \in D$ and $(x, y)$ dominates $(x', y')$ in $H^+$ implies $(x', y') \in D$*
   - *$(x, y), (y, z) \in D$ implies $(x, z) \in D$.*

**Proof.** The equivalence of 1 and 2 is proved in Theorem 2.2. Furthermore, Lemma 2.4 shows that 2 implies 3. It is also quite straightforward to see that 4 implies 2; it suffices to define $a < b$ if $(x, y) \in D$. Thus it remains to show that 3 implies 4.

Therefore, we assume that $H$ has no invertible pair. Note that we may assume that $H$ is weakly connected, otherwise we can order each weak component separately. We also note that for each strong component $C$ of $H^+$, there is a corresponding reversed strong component $C'$ whose pairs are precisely the reversed pairs of the pairs in $C$; we shall say that $C, C'$ are *coupled* strong components.

The partition of $V(H^+)$ into $D, D'$ will correspond to separating each pair of coupled strong components $C, C'$ of $H^+$. The vertices of one strong components will be placed in the set $D$, their reversed pairs will go to $D'$. We wish to make these choices in such a way as to avoid creating a *circular chain* in $D$, i.e., a sequence of pairs $(x_0, x_1), (x_1, x_2), \ldots,$ $(x_n, x_0) \in D$.

We shall proceed as follows. Initially the sets $D$ and $D'$ are empty. We say that a strong component $C$ of $H^+$ is *ripe* when it has no edge *to* another strong component in $H^+ - D$. In the general step, we shall take a ripe component $C$ and place it in $D$, and simultaneously place $C'$ in $D'$. (Note that $C'$ need not be ripe, but has no edge *from* another strong component.) We will show that there is always at least one ripe strong component which can be added to $D$ without creating a circular chain.

The sets $D, D'$ will always have the following properties (which are true initially). There is no circular chain in $D$; each strong component of $H^+$ belongs entirely to $D$, $D'$, or to $V(H^+) - D - D'$; the pairs in $D'$ are precisely the reversed pairs of the pairs in $D$; there is no edge of $H^+$ from $D$ to a vertex outside of $D$; and there is no edge of $H^+$ from a vertex outside of $D'$ to a vertex in $D'$. At the end of the algorithm each pair $(x, y)$ with $x \neq y$ will belong either to $D$ or to $D'$, and hence the final $D$ will have no circular chain and hence satisfy the transitivity property of 4.

We now prove that the algorithm maintains these properties.

Suppose, for a contradiction, that the current $D$ has no circular chain but the addition of $C$ to $D$ creates a circular chain in $C \cup D$. Suppose $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ is a circular chain that has occurred for the first time during the execution of the algorithm, and also suppose that at that time no shorter circular chain has occurred. Since there are no invertible pairs, and since we never place both an edge and its reverse in $D$, we must have $n \geq 2$. We may assume without loss of generality that $(x_n, x_0) \in C$; note that other pairs of the circular chain could also be in $C$.

**Case 1.** Assume that in $H$, there is at least one edge between the vertices $x_0, x_1, \ldots, x_n$, say an edge $x_a x_b$.

We claim that this implies that $H$ is complete on $x_0, x_1, \ldots, x_n$. We make the following elementary observations, assuming $j \neq i$.

1. If $x_j$ dominates $x_i$ then $x_{j-1}$ dominates $x_i$ in $H$.

2. If $x_j$ dominates $x_i$ then $x_j$ dominates $x_{i-1}$ in $H$.

To prove the first observation, we note that if $x_j$ dominates $x_i$ but $x_{j-1}$ not dominate $x_i$ in $H$, then $(x_{j-1}, x_j)$ dominates $(x_{j-1}, x_i)$ in $H^+$. Since $(x_{j-1}, x_j)$ is in $C \cup D$, the pair $(x_{j-1}, x_i)$ must belong to $C \cup D$, implying a shorter circular chain in $C \cup D$.

To prove the second observation, we similarly note that if $x_j$ dominates $x_i$ but $x_j$ does not dominate $x_{i-1}$ in $H$, then $(x_{i-1}, x_i)$ dominates $(x_{i-1}, x_j)$ in $H^+$, also implying a shorter circular chain.

Consider now the fact that $x_a$ dominates $x_b$ in $H$. Property 1 implies that $x_{a-1}$, $x_{a-2}, \ldots,$ $x_{b+1}$ all dominate $x_b$. Since $x_{b+1}$ dominates $x_b$, property 2 implies that $x_{b+1}$ dominates $x_{b-1}$, $x_{b-2}, \ldots,$ $x_{b+2}$, i.e., dominates all other vertices. At this point we use 1 again to derive that $x_b$ dominates $x_{b-1}$, and repeated application of 2 as before implies that $x_b$ dominates all other vertices. Continuing this way, we see that each $x_j$ dominates all other vertices, i.e., the vertices $x_0$, $x_1, \ldots,$ $x_n$ induce a complete graph in $H$.

We conclude the proof of Case 1 by showing that $C$ is a trivial component (with a single vertex). If $C$ has more than one vertex, then so does its corresponding coupled component $C'$, which contains the vertex $(x_0, x_n)$. Hence we assume for contradiction that $(x_0, x_n)$ dominates some $(a, b)$ not in $C \cup D$.

Up to symmetry, we may assume that $x_0$ dominates $a$ in $H$, $x_n$ dominates $b$ in $H$ and $x_0$ does not dominate $b$ in $H$. Since $(a, b)$ is not in $C \cup D$, the pair $(x_0, x_1)$, which is in $C$, cannot dominate $(a, b)$, which implies that $x_1$ does not dominate $b$ in $H$. If $x_2$ dominates $b$ in $H$, then $(x_1, x_2)$ dominates $(x_0, b)$ which dominates $(a, b)$ in $H^+$; this is impossible, as this is a directed path starting in $C$ and ending outside of $C \cup D$, so some edge would exit from $C \cup D$ against the rules we maintain. Therefore $x_2$ does not dominate $b$ in $H$; if $x_3$ dominates $b$ in $H$, then $(x_2, x_3)$ dominates $(x_1, b)$ which dominates $(x_0, b)$ which dominates $(a, b)$, yielding the same contradiction. Therefore $x_3$ does not dominate $b$ in $H$, and continuing this way we would derive that $x_n$ does not dominate $b$, which is false.

Thus we have $C = \{(x_n, x_0)\}, C' = \{(x_0, x_n)\}$. The same proof also shows that $C'$ is ripe, as no $(a, b)$ dominated by $(x_0, x_n)$ can exist outside of $C \cup D$. It is now easy to see that if both $(x_n, x_0)$ and $(x_0, x_n)$ complete a circular chain with $D$, then $D$ already had a circular chain.

**Case 2.** Assume that vertices $x_0, x_1, \ldots, x_n$ are independent in $H$.

**Lemma 3.3** *Suppose $p$ is a vertex of $H$, distinct from $x_0, x_1, \ldots, x_n$, which dominates $x_{i+1}$ but not $x_i$ (or which is dominated by $x_{i+1}$ but not by $x_i$).*

*Then $(x_0, x_1), \ldots, (x_i, p), (p, x_{i+2}), \ldots, (x_n, x_0)$ is also a circular chain created at the same time.*

**Proof.** We conclude from the assumption that $(x_i, x_{i+1})$ dominates $(x_i, p)$ in $H^+$, and since $(x_i, x_{i+1})$ is in $C \cup D$, we must also have $(x_i, p)$ in $C \cup D$. Furthermore, since $x_{i+1}$ does not dominate or is dominated by $x_{i+2}$ in $H$, we also have $(x_{i+1}, x_{i+2})$ dominating $(p, x_{i+2})$, whence $(p, x_{i+2})$ is in $C \cup D$. In conclusion, we see that any such vertex $p$ can replace $x_{i+1}$ in the circular chain $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$. $\square$

**Lemma 3.4**

- *If $p$ is a vertex of $H$, distinct from $x_0, x_1, \ldots, x_n$, which dominates $x_j$ and $x_k$ with $j \neq k$, then $p$ dominates each $x_i$.*
- *If $p$ is a vertex of $H$, distinct from $x_0, x_1, \ldots, x_n$, which is dominated by $x_j$ and $x_k$ with $j \neq k$, then $p$ is dominated by each $x_i$.*
- *If $p$, distinct from $x_0, x_1, \ldots, x_n$, dominates $x_j$ and is dominated by $x_k$ with $j \neq k$, then $p$ both dominates and is dominated by each $x_i, i \neq j, k$.*

**Proof.** If $p$ dominates $x_{i+1}$ but not $x_i$, then Lemma 3.3 implies that $p$ can replace $x_{i+1}$ in the circular chain; however at least one of $x_j, x_k$ is not equal to $x_{i+1}$, whence the vertices of the chain are not independent and we conclude by Case 1. The other items are proved similarly. $\square$

We now claim that the circular chain $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ has at most one pair, say $(x_n, x_0)$, in $C$ (with all other pairs in $D$). Otherwise, assume some $(x_i, x_{i+1}), i \neq n$ is also in the strong component $C$, and let $P$ be a directed path in $C$ from $(x_n, x_0)$ to $(x_i, x_{i+1})$. Let the penultimate pair on this path be $(p, q)$, and, without loss of generality, assume that $px_i, qx_{i+1} \in E(H), px_{i+1} \notin E(H)$. (In the case $x_ip, x_{i+1}q \in E(H), x_{i+1}p \notin E(H)$, the argument is symmetric.) By Lemma 3.3, $p$ does not dominate any $x_j$ with $j \neq i$. Next we claim that $q$ does not dominate $x_i$. Indeed, if $q$ dominates $x_i$, then Lemma 3.4 implies that $q$ dominates all $x_j$. This is a contradiction, since it would mean that $(p, q)$ dominates $(x_i, x_{i+2})$ in $H^+$, implying that $(x_i, x_{i+2})$ is in $C \cup D$ and thus there is a shorter circular chain in $H$. Therefore $q$ does not dominate $x_i$. By a double application of Lemma 3.3, we conclude that we can replace $x_i$ and $x_{i+1}$ by $p$ and $q$ in the circular chain in $H$. Continuing this way, we replace $(p, q)$ by the previous pair on the path $P$, until we obtain the pair $(p', q')$ which is the first pair after $(x_n, x_0)$. Since $x_0$ is adjacent to $q'$, we are back in Case 1.

Thus the circular chain $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ has only the pair $(x_n, x_0)$ in $C$, and any circular chain in $C \cup D$ has exactly one pair in $C$. We now suppose, in addition to the previous assumptions, that our circular chain minimizes the sum of the lengths of all distances amongst the vertices $x_0, x_1, \ldots, x_n$, in the underlying graph of $H$ .

The digraph $H$ turns out to have a very special structure. We claim that in this situation there exists a non-empty set $K$ of vertices of $H$ such that $H \setminus K$ has weak components $C_1, C_2, \ldots C_m$, where $x_i \in C_i, i = 1, 2, \ldots, n$, and such that if $p \in K$ dominates (respectively is dominated by) a vertex in $C_i$, then $p$ dominates (respectively is dominated by) all vertices in $C_i$; moreover, if $x'_0$, $x'_1$, $\ldots, x'_n$ are any vertices with $x'_i \in C_i$, then $(x'_0, x'_1)$, $(x'_1, x'_2), \ldots, (x'_n, x'_0)$ is also a circular chain.

Indeed, we let $K$ consist of all vertices of $H$ that dominate each $x_i$, or are dominated by each $x_i$. It is easy to see that $K$ must be non-empty, as Lemma 3.4 implies that any $p$ dominated by $x_j, x_k, j \neq k$ belongs to $K$. Such a $p$ must exist by our new minimality assumption, as otherwise we could replace $x_j$ by its neighbour $p$ on a path joining $x_j$ to $x_k$ by Lemma 3.3.

The same argument shows that two different $x_j, x_k$ cannot lie in the same weak component $C_i$ of $H \setminus K$, as any path joining $x_j$ to $x_k$ was shown to contain a vertex of $K$. Therefore we can number the components so that $C_i$ contains $x_i$ for $i = 1, 2, \ldots n$. (There may be additional components $C_i$ with $i = n + 1, \ldots, m$.) Now Lemma 3.3 implies that each $x_i$ can be replaced by any neighbour in $C_i$, thus any vertex of $C_i$ can be taken as $x_i$. Thus each $p \in K$ that dominates a vertex in $C_i$ also dominates all vertices in $C_i$, and similarly for vertices $p$ dominated by a vertex in $C_i$.

This creates a situation where any pair $(y, y')$ in the strong component $C$ of $H^+$ containing $(x_n, x_0)$ must satisfy $y \in C_n, y' \in C_0$. This easily implies that the strong component $C$ does not have any arcs entering it from the outside, and hence the strong component $C'$ coupled with $C$ is also ripe. We claim that $C'$ cannot complete a circular chain with $D$. Otherwise, the pair $(x_0, x_n)$ would also complete a circular chain by the same argument. Thus both $(x_0, x_n)$ and $(x_n, x_0)$ complete a circular chain with $D$, whence $D$ must already contain a circular chain, a contradiction.

Of course, if the addition of $C'$ does not create a circular chain, then we add $C'$ to $D$ and $C$ to $D'$. $\square$

This gives us a polynomially verifiable forbidden subgraph characterization of adjusted interval digraphs. As noted above, checking for invertible pairs amounts to computing the strong components of $H^+$ and checking for the existence of a pair $(u, v), (v, u)$ in one strong component. Thus recognition of adjusted interval digraphs is polynomial.

**Corollary 3.5** *Let $H$ be a reflexive digraph. Then $H$ is an adjusted interval digraph if and only if it has no invertible pair.* $\diamond$

We leave the following open problems related to adjusted interval digraphs.

1. Find a linear time recognition algorithm.
2. Find intractable digraph problems that can be solved in polynomial time on the class of adjusted interval digraphs.

## 4 Polymorphisms and the List Homomorphism Problem

The min orderings defined above are a particular case of the following general concept. Let $k$ be a positive integer. The *k-th power* of $H$ is the digraph $H^k$ with vertex set $V(H)^k$ in which $(u_1, u_2, \ldots, u_k)(v_1, v_2, \ldots, v_k)$ is an edge just if each $u_iv_i$ is an edge of $H$. A *polymorphism of order* $k$ is a homomorphism of $H^k$ to $H$. A polymorphism $f$ is *conservative* if $f(u_1, u_2, \ldots, u_k)$ always is one of $u_1, u_2, \ldots, u_k$. From now on we shall use the word *polymorphism to mean a conservative polymorphism.*

A polymorphism $f$ of order two is *commutative* if $f(u, v) = f(v, u)$ for any $u, v$. If $H$ admits a min ordering $<$, then clearly defining $f(u, v) = \min(u, v)$ is a polymorphism, which is commutative.

Two ternary polymorphisms also play a role in the problems LHOM($H$) [2]. A polymorphism $f : H^3 \to H$ is called a *majority polymorphism* if $f(u, u, v) = f(u, v, u) = f(v, u, u) = u$ for any $u, v$. A ternary polymorphism $f : H^3 \to H$ is called a *Maltsev polymorphism* if $f(u, u, v) = f(v, u, u) = v$ for any $u, v$. A ternary polymorphism $f : H^3 \to H$ is *majority (respectively Maltsev) over* $a, b$, if $f(a, a, b) = f(a, b, a) = f(b, a, a) = a, f(b, b, a) = f(b, a, b) = f(a, b, b) = b$ (respectively if $f(a, a, b) = f(b, a, a) = b, f(a, b, b) = f(b, b, a) = a$).

At this point, we can state the classification of LHOM($H$) due to Bulatov. The theorem applies to any relational structure $H$, but for our purposes we only need to state it for reflexive digraphs. Recall that by our definition each polymorphism is conservative. Also, we formulate the result in a language of binary commutative polymorphisms in place of the more usual semi-lattice operations [2], since it is equivalent and is more convenient in our context.

**Theorem 4.1** *[2] Let $H$ be a reflexive digraph.*

*If for every pair of vertices $a, b$ of $H$ there exists a polymorphism of $H$ which is either ternary and majority, or Maltsev, over $a, b$, or is binary and commutative over $a, b$, then LHOM($H$) is polynomial time solvable.*

*Otherwise, if some pair of vertices $a, b$ does not admit any of these polymorphisms, then the problem LHOM($H$) is NP-complete.*

The following fact follows directly from Theorems 2.3 (or Theorem 4.1) and 2.2.

**Theorem 4.2** *If $H$ is an adjusted interval digraph, then LHOM($H$) is polynomial time solvable.*

Here is an equivalent form of a conjecture from [10, 16].

**Conjecture 4.3** *If a digraph $H$ is not an adjusted interval digraph, then LHOM($H$) is NP-complete.*

We also had a similar conjecture for irreflexive digraphs [10, 16]. However, that conjecture has turned out to be false [17, 3], and we shall discuss the case of irreflexive digraphs in [17].

We are currently working, with C. Carvalho, on an algebraic approach to Conjecture 4.3, suggested by [3].

# 5 Some intractable cases

We first collect some available information about known intractable cases of LHOM($H$) for reflexive digraphs $H$.

**Theorem 5.1** *Let $H$ be a reflexive digraph. If*

- *$H$ contains the directed three-cycle $\vec{C}_3$, or*
- *$U(H)$ contains a chordless cycle of length greater than three, or*
- *$S(H)$ is not an interval graph,*

*then the problem LHOM($H$) is NP-complete.*

**Proof.** The problem LHOM($\vec{C}_3$) is shown NP-complete in [10]. (All other problems LHOM($H$) for reflexive digraphs with up to three vertices are known to be polynomial time solvable [10].) The NP- completeness of LHOM($\vec{C}_3$) also follows from Theorem 6.1 and the remark following it.

If $U(H)$ contains a chordless cycle of length greater than three, then even a very special list homomorphism problem (the so-called "retraction problem" RET($H$)) is NP-complete, see [9, 22]. This implies that the more general problem LHOM($H$) is also NP-complete.

If $S(H)$ is not an interval graph then the undirected graph problem LHOM($S(H)$) is NP-complete by [6]. Since an undirected instance $G$ of LHOM($S(H)$) can be viewed as a directed graph with each edge symmetric, this implies that LHOM($H$) is also NP-complete. □

To complete the picture, we shall now show that we may restrict our attention to digraphs $H$ for which both $S(H)$ and $U(H)$ are interval graphs.

**Theorem 5.2** *If $U(H)$ is not an interval graph, then the problem LHOM$(H)$ is NP-complete.*

**Proof.** If $U(H)$ is not an interval graph, then by the theorem of Lekkerkerker and Boland [23], $U(H)$ must contain a chordless cycle of length greater than three, or an asteroidal triple. In view of the last theorem, we may assume that $U(H)$ contains an asteroidal triple, i.e., a triple of vertices $0, 1, 2$ and paths $P(0, 1), P(0, 2), P(1, 2)$ (where $P(i, j)$ joins vertices $i$ and $j$), such that each vertex $i$ from $0, 1, 2$ has no neighbours on the path joining the other two vertices.

We first recall gadgets called choosers from [6, 21], as discussed in [20]. We state the definition in a slightly more general form, and apply it to digraphs. Let $i, j$ be distinct vertices from $0, 1, 2$ and let $I, J$ be subsets of $\{0, 1, 2\}$. A *chooser* $Ch(i, I; j, J)$ is a digraph $P$ with specified vertices $a$ and $b$, and with lists $L(p) \subseteq V(H)$, for $p \in V(P)$, such that any list homomorphism $f$ of $P$ to $H$ has $f(a) = i$ and $f(b) \in I$ or $f(a) = j$ and $f(b) \in J$; and for any $i' \in I$ and $j' \in J$ there is a list homomorphism $f$ of $P$ to $H$ with $f(a) = i$ and $f(b) = i'$ and a list homomorphism $g$ of $P$ to $H$ with $g(a) = j$ and $g(b) = j'$.

It is shown in [6], as explained in [20] page 174-5, that if there exist choosers $Ch(i, \{i, k\}; j, \{j, k\})$ and $Ch(i, \{i\}; j, \{k\})$, for any permutation $ijk$ of $0, 1, 2$, then $LHOM(H)$ is NP-complete. (Those proofs are stated in terms of undirected graphs $H$ and choosers $Ch$ that are paths, but they apply verbatim to arbitrary digraph choosers $Ch$ as defined here.)

These choosers will be constructed from simpler building blocks which we call separators. A *separator* $G(i), i = 0, 1, 2$, is a digraph with two specified vertices $u, v$ and lists $L(t), t \in V(G(i))$, such that

- every list homomorphism of $G(i)$ to $H$ with respect to the lists $L$ maps both $u, v$ to $i$ or maps neither of $u, v$ to $i$, and
- for any pair of values $x, y$ from $0, 1, 2$ in which neither or both values $x, y$ are equal to $i$, there is a list homomorphism of $G(i)$ with respect to the lists $L$, mapping $u$ to $x$ and $v$ to $y$.

The proof will be completed by the following two lemmas. □

**Lemma 5.3** *If there exists a separator $G(i)$ for each $i = 0, 1, 2$, then the problem $LHOM(H)$ is NP-complete.*

**Proof.** The separators can be used to construct the choosers as follows: $Ch(i, \{i\}; j, \{k\})$ is formed from $G(i)$ by setting $a = u$ and modifying its list to $L(u) = \{i, j\}$, and by setting $b = v$ and modifying its list to $L(v) = \{i, k\}$. To form $Ch(i, \{i, k\}; j, \{j, k\})$, we take four vertices $a, b, c, d$, and place one copy of $G(i)$ between $a$ and $c$ (identifying $a$ with $u$ and $c$ with $v$), and another copy of $G(i)$ between $b$ and $d$ (identifying in a similar manner), as well as a copy of $G(j)$ between $c$ and $b$ and another copy of $G(j)$ between $d$ and $a$. It is easy to check that the resulting digraph satisfies the conditions for a chooser $Ch(i, \{i, k\}; j, \{j, k\})$ with the specified vertices $a$ and $b$. $\square$

**Lemma 5.4** *If the $U(H)$ has an asteroidal triple, then $H$ has separators $G(i), i = 0, 1, 2$.*

**Proof.** Suppose $H$ has $n$ vertices, and $U(H)$ has an asteroidal triple $0, 1, 2$. The separator $G(i)$ will be an oriented path of length $2n$, with alternating forward and backward edges. The lists of the two end vertices of the path $G(i)$ are $\{0, 1, 2\}$. All other vertices of $G(i)$ have lists consisting of $i$, together with all the vertices on the path $P(j, k)$ in $H$ (from the definition of an asteroidal triple). Note that the length of the path $G(i)$ and the orientation of its edges ensure that it admits a homomorphism (without considering the lists) that maps $u$ and $v$ to any two vertices of $H$ (recall that every vertex has a loop). It follows from the definition of an asteroidal triple that any list homomorphism of $G(i)$ to $H$ maps both $u$ and $v$ to $i$, or neither of $u, v$ to $i$; and moreover, that there are list homomorphisms of $G(i)$ to $H$ mapping both $u$ and $v$ to $i$ and both to $j, k$ in any prescribed combination, i.e., that $G(i)$ is a separator. $\square$

# 6 Trees and Semi-Complete Digraphs

We now verify Conjecture 4.3 in the important cases. Recall that Theorem 5.2 implies that it suffices to focus on digraphs $H$ such that $U(H)$ is an interval graph. All complete graphs and certain trees (that is, caterpilars) are in some sense the most basic interval graphs, and that is where we turn next.

A digraph is *semi-complete* if its underlying graph is complete. A digraph is a *tree* if its underlying graph (loops and parallel edges ignored) is a tree in the usual sense.

**Theorem 6.1** *Suppose $H$ is a reflexive semi-complete digraph. If $H$ contains an invertible pair, then LHOM($H$) is NP-complete.*

**Proof.** We will appeal to Bulatov's characterization, Theorem 4.1, showing that if there exist invertible pairs in $H$, then some invertible pair $a, b$ admits no polymorphism as prescribed by Theorem 4.1.

It turns out that some structures in $H$ limit our choices of polymorphisms from the theorem. Let $R$ be the reflexive digraph $V(R) = \{a, b, c\}$ and $E(R) = \{aa, bb, cc, ab, bc, ac, ca\}$.

**Lemma 6.2** *There is no polymorphism $g$ on the digraph $R$ which is a majority over $a, b$.*

**Proof.** Suppose $g$ is a polymorphism of $R$ which is a majority over $a, b$, i.e., $g(a, a, b) = g(a, b, a) = g(b, a, a) = a$, and $g(a, b, b) = g(b, a, b) = g(b, b, a) = b$. We claim that $g$ must also be a majority over $b, c$. Note that $g(c, c, b)g(a, a, b) = g(c, c, b)a \in E(R)$. Hence $g(c, c, b) = c$, as $b$ does not dominate $a$ in $R$. Similarly, $g(c, b, c) = g(b, c, c) = c$. Also $g(b, b, c)g(b, b, a) = g(b, b, c)b \in E(R)$ thus $g(b, b, c) = b$ and similarly $g(b, c, b) = g(c, b, b) = b$. Now we can conclude that $g$ is also majority over $a, c$, using the fact that $g(a, a, c)g(b, b, c) \in E(R)$ and $g(b, b, c)g(c, c, a) \in E(R)$.

Now we note that we have $g(a, b, c)g(b, b, c) = g(a, b, c)b \in E(R)$, which implies that $g(a, b, c) \in \{a, b\}$ (since $c$ doesn't dominate $b$ in $R$); we have $g(a, b, b)g(a, b, c) \in E(R)$, which similarly implies that $g(a, b, c) \in \{b, c\}$; and we have $g(c, a, c)g(a, b, c) \in E(R)$, which similarly implies that $g(a, b, c) \in \{a, c\}$, which is impossible. □

**Lemma 6.3** *Suppose $H$ is a reflexive digraph with $ab \in E(H), ba \notin E(H)$. There is no polymorphism $h$ over the digraph $H$ which is a Maltsev operation over $a, b$.*

**Proof.** If $h$ is Maltsev over $a, b$, then $h(a, a, b)h(a, b, b) = ba \in E(H)$, a contradiction. □

Thus let us assume $H$ contains invertible pairs. If $H$ also contains an induced reflexive directed three-cycle $\vec{C}_3$, then LHOM($H$) is known to be NP-complete [10]. Thus we may assume for the proof that $H$ does not contain $\vec{C}_3$. By a similar argument, we may assume that $S(H)$ is an interval graph, and in particular, $S(H)$ does not contain an induced four-cycle [9, 22].

If $H$ contains invertible pairs, then there exist directed closed walks $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), (x_0, y_0)$ in $H^+$ which contains both $(a, b)$ and $(b, a)$ for some $a, b \in V(H)$. We say that such a closed walk $C$ is an *inverting walk* for the pair $a, b$. As noted in Lemma 2.1, each vertex $(x_i, y_i)$ of $C$ is itself invertible.

An inverting walk $C$ in $H^+$ consists of forward edges only. Recall that, in $H$, these edges could correspond to pairs of edges $x_i x_{i+1}$, $y_i y_{i+1}$, which are either forward or backward.

We first assume that for some $C$ and some $i$ we have the edges $x_i x_{i+1}, x_{i+2} x_{i+1} \in E(H)$ and $y_i y_{i+1}, y_{i+2} y_{i+1} \in E(H)$. Without loss of generality, let us assume $i = 0$, i.e., that $x_0 x_1, x_2 x_1$, $y_0 y_1, y_2 y_1 \in E(H)$ and $x_0 y_1, y_2 x_1 \notin E(H)$. Therefore, the pair $(x_0, y_1)$ dominates $(x_1, y_1)$ and is dominated by $(x_0, y_0)$, which are consecutive in the cycle $C$. Thus we may assume that $(x_0, y_1)$ is also in $C$, and hence is invertible. The same argument shows that $(x_1, y_2)$ is also invertible.

Since $H$ is semi-complete, we must have $y_1 x_0, x_1 y_2 \in E(H)$. If $y_1 x_1 \notin E(H)$, then $y_1, x_0, x_1$ are all distinct and must induce $R$, since there is no induced $\vec{C}_3$). Then over

$y_1, x_0$ there is no majority by Lemma 6.2, no Maltsev by Lemma 6.3, and no commutative binary polymorphism by Lemma 2.4. Hence LHOM($H$) is NP-complete by Theorem 4.1.

If $y_1x_1 \in E(H)$, then $y_1, x_1, y_2$ must be distinct and the same argument as above implies that $x_1y_1 \in E(H)$. Then the same argument again applied to the triple $y_1, x_0, x_1$ implies that $x_1x_0 \in E(H)$, and applied to the triple $y_1, x_1, y_2$ implies that $y_1y_2 \in E(H)$. Note that $x_0 \neq y_2$ because $x_0y_1 \notin E(H)$ but $y_2y_1 \in E(H)$. If $y_2x_0 \notin E(H)$ then we have a copy of $R$ over $x_0, y_1, y_2$; if $x_0y_2 \notin E(H)$ then we have a copy of $R$ over $x_0, x_1, y_2$. This yields an induced four-cycle $x_0, x_1, y_1, y_2, x_0$ in $S(H)$, contrary to our assumption.

Thus we may assume that on any inverting walk all edges go in the same direction, forward or backward. Without loss of generality, assume that on $C$ all edges $x_ix_{i+1}$ in $H$ are forward (and similarly for $y_iy_{i+1}$). If there is a copy of $\vec{C}_3$ or $R$, the problem LHOM($H$) is NP-complete as above. Otherwise, we claim that all $x_iy_i \in E(H)$ and $y_ix_i \in E(H)$. Indeed if $y_ix_i \notin E(H)$, then a copy of $\vec{C}_3$ or $R$ exists on $x_{i-1}, x_i, y_i$, unless $x_i = x_{i-1}$. Note that if $x_i = x_{i-1}$ would mean that $x_iy_i \notin E(H)$ also holds, contrary to the fact that $H$ is semi-complete. If $x_iy_i \notin E(H)$, then on some inverting walk involving the invertible pair $x_i, y_i$, the same argument would show the existence of $\vec{C}_3$ or $R$. □

Thus the conjecture holds for semi-complete digraphs. We now turn to trees.

It is well known [13] that a tree is an interval graph if and only if it is a caterpillar, i.e., if the removal all leaves yields a path. Thus we want to decide which orientations of caterpillars yield adjusted interval digraphs. Let $S(x)$ denote the set of leaves of $H$ adjacent to the vertex $x \in P$. As usual, we refer to $H$ as a tree, or star, etc., to mean that $U(H)$ (without the loops) is a tree, or star, etc., respectively.

We begin by stating a more convenient definition of a min ordering for reflexive digraphs.

**Lemma 6.4** *Let $H$ be a reflexive digraph. Then an ordering $<$ of $V(H)$ is a min ordering if and only if for any three vertices $i < j < k$ we have*

- $ik \in E(H)$ *implies* $ij \in E(H)$, *and*
- $ki \in E(H)$ *implies* $ji \in E(H)$.

**Proof.** The necessity of the two properties follows by taking the edge $ik$ (respectively $ki$) and the loop at $j$. To see the sufficiency, consider edges $xy, x'y'$ of $H$ and assume without loss of generality that $x < x', y' < y$; thus $\min(x, x') \min(y, y') = xy'$. If $x = y'$, then $xy'$ is an edge since $H$ is reflexive. If $x < y'$, then $xy'$ is an edge because of the triple $x < y' < y$. If $y' < x$, then $xy'$ is an edge because of the triple $y' < x < x'$. □

**Corollary 6.5** *Let $H$ be a reflexive digraph. An ordering of the vertices of $H$ is a min ordering if and only if for each vertex $v$ the vertices that follow $v$ in the ordering consist of*

1. *first, all vertices that are adjacent to $v$ by double edges,*
2. *second, all vertices that are adjacent to $v$ by single edges, either all forward or all backward, and*
3. *last, all vertices that have no edges to or from $v$.*

Of course, any of the three groups could be empty. Note that, in particular, in a min ordering of $H$ it cannot be the case that a vertex $v$ has both forward and backward edges towards vertices that follow it in the ordering.

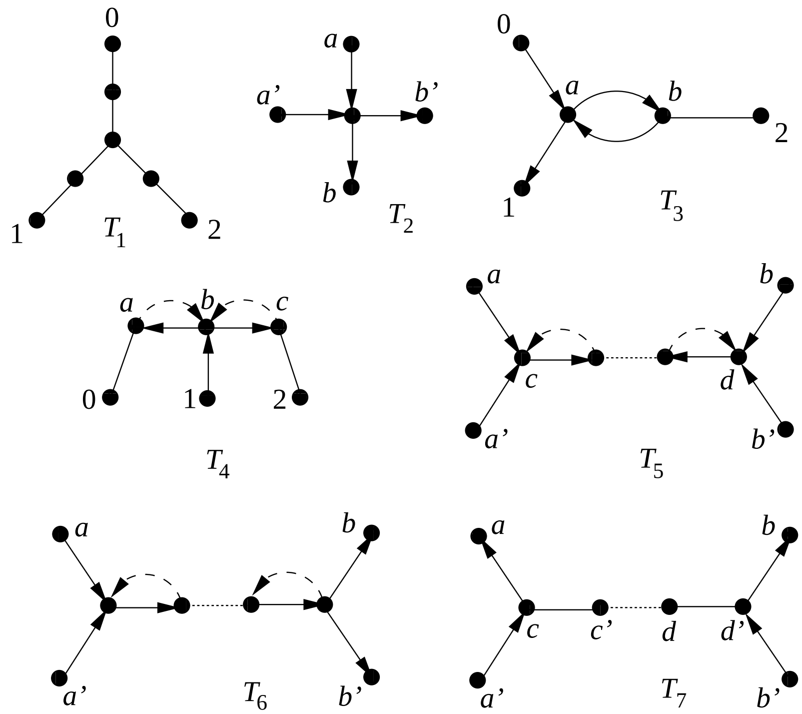
If $H$ is a star, we shall define $H$ to be a *good caterpillar*, if it does not contain, as induced subgraph, the tree $T_2$ depicted below. If $H$ is not a star, we define it to be a *good caterpillar* if it has a longest path $v_0, v_1, \ldots, v_k, v_{k+1}$ satisfying the following conditions for all $i$. (Note that $v_1, v_2, \ldots, v_k$ is the path $P$, and that $v_0 \in S(v_1), v_{k+1} \in S(v_k)$.)

1. If $v_i v_{i+1} \in E(H)$, then $v_i v \in E(H)$ for all $v \in S(v_i) - v_{i-1}$.
2. If $v_{i+1} v_i \in E(H)$, then $v v_i \in E(H)$ for all $v \in S(v_i) - v_{i-1}$.

Note that if $v_i v_{i+1}$ is a double edge then so are all $v_i v, v \in S(v_i) - v_{i-1}$. Observe that there are no restrictions on $v_0$, other than those arising from the restrictions on $v_1$. Indeed, all edges $v_1 v$ for $v \in S(v_1) - v_0$ must follow the direction of the edge $v_1 v_2$ (forward, backward, or double) - with the possible exception of a single vertex $v$, which must be the vertex $v_0$. Thus such a $v_0$ can be chosen if and only if the restrictions on $v_1$ have at most one exception. Similarly, there are no restrictions on $v_{k+1}$, other than those arising from the restrictions on $v_k$. All edges $v_k v$ for $v \in S(v_k)$ must follow the direction of the edge $v_k v_{k+1}$. It is easy to see that such a $v_{k+1}$ can be chosen if and only if between $v_k$ and $S(v_k)$ there does not exist at the same time a single forward and a single backward edge. Finally, we note that the exceptional case, when $H$ is a star, also conforms to the general definition; we have chosen to state it separately only for convenience.

**Theorem 6.6** *Let $H$ be a reflexive digraph that is a tree. Then the following statements are equivalent.*

1. *$H$ is a good caterpillar*
2. *$H$ is an adjusted interval digraph*




*3. H does not contain (as an induced subgraph) any of the trees $T_1, \ldots T_7$ or their reverses.*

**Proof.** The edges in the trees $T_1, \ldots T_7$ that are not oriented can be forward, backward, or double; the dashed edges are optional.

We shall show that 1 implies 2, 2 implies 3, and 3 implies 1. Indeed, 1 implies 2 via Theorem 2.2, as a good caterpillar can be ordered starting from $v_0$ and proceding to $v_1, v_2, \ldots, v_k$, with listing the double edges of $S(v_i) - v_{i-1}$ first, as suggested by Corollary 6.5. The definition of a good caterpillar ensures that the listing for $S(v_i) - v_{i-1}$ can be chosen to end with $v_{i+1}$.

Theorem 2.2 also allows us to derive 3 from 2: none of the forbidden subtrees allows a min ordering. To see this, in the trees $T_1, T_3, T_4$ focus on the vertices $0, 1, 2$, and on the trees $T_2, T_5, T_6, T_7$ focus on the vertices $a, a', b, b'$.

It remains to show that 3 implies 1. Thus suppose $H$ is a reflexive tree which does not contain any of $T_1 - T_7$ or their reverses. Since $H$ does not contain $T_1$ it is a caterpillar. If $H$ is a star, the conclusion now follows. Thus assume $H$ is not a star: when all leaves of $H$ are removed we obtain a path $P$, say $P = p, r, s, \ldots, y, z$. We will prove that one of $p, z$ can be chosen as $v_1$ and the other as $v_k$. Suppose first that $p$ cannot be chosen to satisfy the condition for $v_1$. Then in $S(p)$ there must be two vertices $v, v'$ such that the edges $pv, pv'$ do not follow the direction of the edge $pr$ on $P$. If $pr$ is a double edge, this means that $pv, pv'$ are single edges. Since $H$ does not contain $T_3$, both are forward (or both backward) edges. This implies that all edges $pv, v \in S(p)$ follow the direction of $pr$, and thus $p$ can be chosen to satisfy the condition for $v_k$. Similarly, if $pr$ is a single (forward or backward edge), $p$ can be chosen as $v_k$, since $H$ does not contain $T_2$. Therefore, each of $p, z$ satisfies the condition for $v_1$ or for $v_k$. Suppose next that neither $p$ nor $z$ satisfy the condition for $v_1$. Then each contains two single edges whose direction does not follow the direction of $pr$; this contradicts the fact that $H$ does not contain $T_5$ and $T_6$ or their reverses. Similarly, the absence of $T_7$ implies that at least one of $p, z$ satisfies the condition for $v_k$. The absence of $T_4$ (and its reverse) implies that each intermediate vertex $r, s, \ldots, y$ of $P$ satisfies the condition for $v_i$ if its left or its right neighbour on $P$ plays the role of $v_{i+1}$. Finally, if one vertex of $P$ requires its left neighbour, while another requires its right neighbour, we again obtain a contradiction as above with the fact that $H$ does not contain the trees $T_5, T_6, T_7$. □

We now recall and enhance the *indicator construction* from [19], cf. [20]. For a fixed *indicator* $I, i, j$ (that is a digraph $I$ with two specified vertices $i, j$) in which each vertex $v$ has a list $L(v) \subseteq V(H)$, the indicator construction transforms a digraph $H$ into the digraph $H^*$, with the same vertex set as $H$, and with adjacency defined by the following rule: $xy$ is an edge of $H^*$ just if there exists a list homomorphism of $I$ to $H$ that maps $i$ to $x$ and $j$ to $y$. It is easy to see that the following extension of Lemma 5.5 of [20] holds. (The proof is identical, with the obvious addition of lists; note that we assume that the lists of the vertices $i$ and $j$ are the entire set $V(H^*)$: this ensures that the proof in [20] properly applies to reduce $LHOM(H^*)$ to $LHOM(H)$).

**Lemma 6.7** *If the problem $LHOM(H^*)$ is NP-complete, and if $L(i) = L(j) = V(H^*)$, then the problem $LHOM(H)$ is also NP-complete.*

We now apply these tools to prove the following dichotomy.

**Corollary 6.8** *Let $H$ be a reflexive digraph that is a tree.*

*If $H$ is a good caterpillar, then $LHOM(H)$ is polynomial time solvable. Otherwise, $LHOM(H)$ is NP-complete.*

**Proof.** If $H$ is a good caterpillar, the theorem implies that it has a min ordering and hence $LHOM(H)$ is polynomial time solvable. Otherwise, the theorem implies that $H$ contains $T_1, T_2, \ldots,$ or $T_7$.

If $H$ contains $T_1$, then $S(H)$ is not an interval graph and hence $LHOM(H)$ is NP-complete by Theorem 5.1.

If $H$ contains $T_2$, then we shall apply Lemma 6.7. Consider the indicator $I$ consisting of three vertices $i, c, j$ and two edges $ic, cj$, with the lists $L(i) = L(j) = \{a, a', b, b'\}$, $L(c) = V(H)$. It is clear that $H^*$ is a reflexive digraph that is a cycle with four vertices. Thus $LHOM(H^*)$ is NP-complete by Theorem 5.1, and $LHOM(H)$ is NP-complete by Lemma 6.7.

If $H$ contains $T_3$ then consider the three vertices $0, 1, 2$ of $T_3$. We shall prove that $LHOM(H)$ is NP-complete using Lemma 5.3. Indeed, since there is a path joining $0, 1$ that avoids the neighbours of 2, the separator $G(2)$ is constructed as in Lemma 5.4. To construct $G(1)$, we take a path that begins with a forward and then a double edge, followed by a sufficiently long alternating sequence of forward and backward edges, and ending with a double edge followed by a backward edge. The lists will be $\{0, 1, 2\}$ everywhere except $a$ will be added to the lists of the second and second to last vertex and $b$ will be added to the third and third to last vertex. This pattern of edges and lists ensures that there is a list homomorphism mapping the first vertex of $G(1)$ to 0 and the last vertex to 2 and conversely, while if the first or last vertex of $G(1)$ is mapped to 1, the entire path must map to 1. The path $G(0)$ is constructed similarly. By Lemma 5.3, $LHOM(H)$ is NP-complete.

If $H$ contains $T_4$, we proceed similarly. Only $G(1)$ requires an explanation: it is enough to take a sufficiently long path of alternating forward and backward edges with a middle vertex $t$ of indegree zero, and assign the lists $\{0, 1, 2\}$ to the end vertices, the lists $\{0, 1, 2, a, c\}$ to all inner vertices except $t$, and the list $\{0, 1, 2, a, b, c\}$ for the special vertex $t$. It is again easy to check that this pattern of forward and backward edges, together with the lists, ensure the required properties for the separator $G(1)$.

If $H$ contains $T_5$, we shall again use Lemma 6.7. The indicator will be a path $I$ from $i$ to $j$ identical to the path $a, c, \ldots, d, b$ in $T_5$. the lists are $L(i) = L(j) = \{a, a', b, b'\}$ and otherwise $L(x) = \{x, a, a', b, b'\}$. It is easy to check that $H^*$ is a reflexive cycle with four vertices. The proof for $T_6$ is similar.

Consider now the last tree $T_7$. If the edge $cc'$ or $dd'$ is double, $T_7$ contains $T_3$ and hence we are done. Thus we shall assume that $c'c, dd'$ are forward edges. (By relabeling we obtain the case when they are both backward edges; the case when one is forward and the other backwards is different, but the proof is similar.) We again proceed to use Lemma 6.7. The indicator will be a path $I$ from $i$ to $j$ consisting of a path from $i$ to a middle vertex $t$ identical to the path $a, c, c', \ldots, d', b'$ in $T_7$, followed by a path from $t$ to $j$ identical to the path $a', d', d, \ldots, c, b$ in $T_7$. The lists are $L(i) = L(t) = L(j) = \{a, a', b, b'\}$ and $L(x) = V(H)$ otherwise. It is easy to check that $H^*$ is the reflexive cycle with edges $ab, ab', a'b, a'b'$. (The path from $i$ to $t$ ensures the presence of the edges $ab, a'b$ and the path from $t$ to $j$ ensures the presence of the edges $ab', a'b'$.) $\square$

In particular, Conjecture 4.3 holds for trees.

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