# Matrix Partitions of Split Graphs

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**Abstract**

Matrix partition problems generalize a number of natural graph partition problems, and have been studied for several standard graph classes. We prove that each matrix partition problem has only finitely many minimal obstructions for split graphs. Previously such a result was only known for the class of cographs. (In particular, there are matrix partition problems which have infinitely many minimal chordal obstructions.) We provide (close) upper and lower bounds on the maximum size of a minimal split obstruction. This shows for the first time that some matrices have exponential-sized minimal obstructions of any kind (not necessarily split graphs). We also discuss matrix partitions for bipartite and co-bipartite graphs.

*Keywords:* generalized graph colouring, matrix partition, split graphs, minimal obstructions, forbidden subgraphs

## 1. Introduction

The approach to graph partition problems, proposed in [9, 1, 5], and used in this paper, is informed by the following distinction between different partition problems.

There are graph partition problems which may be solved in polynomial time and for which the set of minimal non-partitionable graphs is finite. The *split graphs recognition problem* is a well-known example [8]. On the other

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hand there are partition problems, such as the *bipartition* problem, which may be solved in polynomial time [10], but for which the set of minimal non-partitionable graphs is infinite (in the case of the bipartition problem, these are the odd cycles). Finally, there are numerous $NP$-complete graph partition problems, such as the *3-colouring* problem.

To discuss a broad class of partition problems, we use *patterns* to describe the requirements of the partition. In particular, the patterns we examine specify partition problems in which the input graph's vertices are to be partitioned into independent sets, or cliques, or some combination of independent sets and cliques. Further, we might require that two parts of vertices in the partition be completely adjacent, or completely non-adjacent. Formally, we use *matrices* to describe these patterns.

Let $M$ be a symmetric $m \times m$ matrix over $0, 1, *$. An *$M$-partition* of a graph $G$ is a partition of the vertices of $G$ into parts $P_1, P_2, \ldots, P_m$ such that two distinct vertices in parts $P_i$ and $P_j$ (possibly with $i = j$) are adjacent if $M(i, j) = 1$, and nonadjacent if $M(i, j) = 0$. The entry $M(i, j) = *$ signifies no restriction.

Note that when $i = j$ these restrictions mean that part $P_i$ is either a clique, or an independent set, or is unrestricted, when $M(i, i)$ is 1, or 0, or $*$, respectively. Since some of the parts may be empty, we may assume that none of the diagonal entries of $M$ are asterisks (otherwise a trivial partition always exists). For a fixed matrix $M$, the $M$-partition problem asks whether or not an input graph $G$ admits an $M$-partition.

If a graph $G$ fails to admit an $M$-partition, we say that $G$ is an *$M$-obstruction.* Further, if $G$ is an $M$-obstruction but deleting *any* vertex of $G$ results in an $M$-partitionable graph, then $G$ is a *minimal* $M$-obstruction.

Given a graph $G$ and lists $L(v) \subseteq \{1, \ldots, m\}$, with $v \in V(G)$, the *list $M$-partition* problem asks whether $G$ admits an $M$-partition respecting the lists. That is, an $M$-partition of $G$ such that, for every $v \in V(G)$, the vertex $v$ is placed in a part $P_i$ only if $i \in L(v)$. In this paper, we will focus on the non-list version, and will explicitly refer to the list version when it is discussed.

For any matrix $M$ in this paper, we assume that there are $k$ zero entries and $\ell$ one entries on $M$'s diagonal. By row and column permutations, we may further assume that $M(0, 0) = M(1, 1) = \ldots = M(k, k) = 0$ and $M(k + 1, k + 1) = \ldots = M(k + \ell, k + \ell) = 1$. Since we suppose that $M$ has no diagonal asterisks we have $k + \ell = m$. Let $A$ be the submatrix of $M$ on rows $1, \ldots, k$ and columns $1, \ldots, k$; let $B$ be the submatrix of $M$ on

$$\begin{pmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{pmatrix}$$

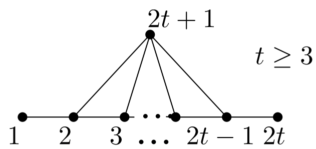


Figure 1.1: A matrix with a family of infinitely many minimal obstructions.

rows $k+1, \dots, m$ and columns $k+1, \dots, m$; and let $C$ be the submatrix of $M$ on rows $1, \dots, k$ and columns $k+1, \dots, m$. We say such an $M$ is in $(A, B, C)$-*block* form.

Feder et al. [7] have shown that if there are asterisks in block $A$ or block $B$ of a matrix $M$, then there are infinitely many minimal $M$-obstructions. Thus, when discussing general graphs, we must restrict our attention to matrices in which the only asterisk entries (if any) are in the block $C$. Such matrices are called *friendly*. Of these, for any $m \times m$ matrix $M$ containing no asterisk entries at all (i.e. having only entries in $\{0, 1\}$), it has been shown that the largest minimal $M$-obstruction is of size $(k+1)(\ell+1)$ [2].

Even when restricted to chordal graphs, there are matrices for which there are infinitely many chordal minimal obstructions [4, 6]. One of these matrices, and an infinite family of chordal minimal obstructions to this matrix, appear frequently in relation to other classes of graphs in this paper, and so are given in Figure 1.1. The obstruction family in this figure is an interval graph, so that the matrix in fact has infinitely many interval minimal obstructions. Nonetheless, for any matrix $M$, the $M$-partition problem restricted to interval graphs can be solved in polynomial time, even with lists [11]. Note that the family in Figure 1.1 is not a family of split graphs, as each member contains $2K_2$ as an induced subgraph.

For general matrices M, all known upper bounds on the size of minimal obstructions to M-partition are exponential [2, 3, 9]; however, in none of these cases has it been shown that exponential-sized minimal obstructions to M-partition actually exist.

This paper is organized as follows: In Section 2, we show that for any $m \times m$ matrix $M$, a split minimal $M$-obstruction has $O(2^m \cdot m^2)$ vertices. This implies that any $M$-partition problem (without lists) is solvable in polynomial time when the input is restricted to split-graphs.

Section 3 exhibits, for a particular class of $m \times m$ matrices, a split minimal obstruction of size $\Omega(2^m/\sqrt{m})$, demonstrating that the exponential upper bound derived in Section 2 is not far from being tight. As noted above, this means that the class of split graph obstructions is the first class with finite minimal obstructions known to contain exponentially large obstructions.

In section 4, we discuss graphs that admit other types of partitions, such as bipartite graphs and co-bipartite graphs. It is shown that for these classes also there only finitely many minimal obstructions for any matrix $M$. These graph classes (including the class of split graphs) have a natural common generalization, namely graphs that are unions of $k$ independent sets and $\ell$ cliques, sometimes called $(k, \ell)$-graphs. Split graphs are $(1, 1)$-graphs, bipartite graphs are $(2, 0)$-graphs, and co-bipartite graphs are $(0, 2)$-graphs. By contrast we show that when $k + \ell \geqslant 3$, there is a matrix $M$ with infinitely many minimal $(k, \ell)$-graph obstructions. When $k \geqslant 2$, there are infinitely many minimal $(k, \ell)$-graph obstructions that are chordal.

## 2. Matrix Partitions of Split Graphs

In this section we prove the following theorem.

**Theorem 2.1.** *If $M$ is a matrix without diagonal asterisks, then there are only finitely many split minimal $M$-obstructions.*

*The size of any split minimal $M$-obstruction is $O(2^m \cdot m^2)$.*

A set of vertices $H \subseteq V(G)$ is said to be *homogeneous* in $G$ if the vertices of $V(G) - H$ can be partitioned into two sets, $S_1$ and $S_2$ such that every vertex of $S_1$ is adjacent to every vertex of $H$, and no vertex of $S_2$ is adjacent to a vertex of $H$. The proof of Theorem 2.1 relies on the existence of large homogenous sets in $M$-partitionable split graphs.

**Proposition 2.2.** *Let $A$ be a $k \times k$ matrix whose diagonal entries are all zero. Let $G_A$ be a split graph that admits an $A$-partition. Then every part $P$ of an $A$-partition of $G_A$ contains a homogeneous set in $G_A$ of size at least $\frac{|P|-1}{2^{k-1}}$.*

*Proof.* Consider an $A$-partition of $G_A$ into parts $P_1, ...P_k$ and a split partition of $G_A$ into a clique $C$ and independent set $I$. Note that for $1 \leqslant i \leqslant k$, we have $|P_i \cap C| \leqslant 1$, since each $P_i$ is an independent set. Moreover, since $I$ is also an independent set, each vertex in the set $P_1 \cap I$ is adjacent to at most $k - 1$ vertices, one in each $P_i \cap C$, for $2 \leqslant i \leqslant k$ (see Figure 2.1). If $P_i \cap C$ contains a vertex, we will denote it by $u_i$. Note that each $u_i$ is either adjacent to at least half of the vertices of $P_1 \cap I$, or non-adjacent to at least half of the vertices of $P_1 \cap I$. Thus there exists a set $X$ of at least $\frac{|P_1|-1}{2^{k-1}}$ vertices of $P_1 \cap I$ that have the same relation (adjacent or non-adjacent) to the each vertex $u_i$. Since $I$ is an independent set, this implies that $X$ is a homogeneous set in $P_1$, and of course the same argument applies to any other $P_j$.

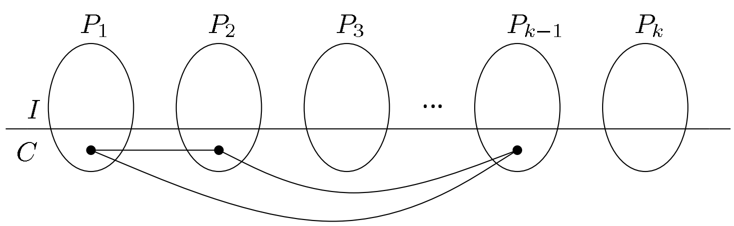


Figure 2.1: Structure of a $k$-partite split graph

□

**Proposition 2.3.** *Let $B$ be an $\ell \times \ell$ matrix whose diagonal entries are all 1. Let $G_B$ be a split graph that admits a $B$-partition. Then every part $P$ of a $B$-partition of $G_B$ contains a homogeneous set in $G_B$ of size at least $\frac{|P|-1}{2^{\ell-1}}$.*

*Proof.* The result follows from Proposition 2.2, since $G_B$ admits a $B$-partition if and only if $\overline{G_B}$ admits a $\overline{B}$-partition, and the complement of a split graph is a split graph. □

We also require the following observation.

**Fact 2.4.** *Let $M$ be an $(A, B, C)$-block matrix and let $G$ be a split graph. If $C$ has an asterisk entry, then $G$ admits an $M$-partition.*

*Proof.* If $C$ has an asterisk, then $M$ contains the matrix $\left(\begin{smallmatrix} 0 & * \\ * & 1 \end{smallmatrix}\right)$ as a principal submatrix. Thus $G$ admits this partition by definition of split graphs, since every other part may be empty. □

*Proof of Theorem 2.1.* Let $M$ be an $m \times m$ matrix, with $k$ diagonal 0s and $\ell$ diagonal 1s. Assume $k \geqslant \ell$. We show that the number of vertices in a split minimal $M$-obstruction is at most

$$2^{k-1}(k + \ell)(2k + 3) + 1 = O(2^k \cdot k^2).$$

Suppose for contradiction that $G$ is a minimal $M$ obstruction with at least $2^{k-1}(k + \ell)(2k + 3) + 2$ vertices. By Fact 2.4, we may assume that the submatrix $C$ has no asterisks. Pick an arbitrary vertex $v$ and consider an $M$-partition of the graph $G-v$. Note that $G-v$ has at least $2^{k-1}(k+\ell)(2k+3)+1$ vertices; as there are $k+\ell$ parts in the partition, by the pigeonhole principle there is a part $P$ of size at least $2^{k-1}(2k+3)+1$. This part $P$ is either an independent set or a clique, and each of these cases will be considered separately below. Either way, by Propositions 2.2 and 2.3, $P$ contains a homogeneous set in $A$ or $B$ (depending on whether $P$ is an independent set or a clique) of size at least $\frac{|P|-1}{2^{k-1}} \geqslant 2k+3$. Since $C$ has no asterisks, this set is homogeneous in $G$. Thus $G-v$ has a homogeneous set of size at least $2k+3$, and so $G$ has a homogeneous set $H$ of size at least $k+2$, since by the pigeonhole principle at least $k+2$ of the vertices of $P$ agree on $v$. Recall that $P$ (and hence the homogeneous set $H$ in it) is an independent set or a clique. Now let $w \in H$, and consider an $M$-partition of $G-w$.

**Case 1.** If $P$ is an independent set, then so is $H$; hence, there are at least $k+1$ independent vertices in $G-w$. As there are $\ell \leqslant k$ clique parts in the partition of $G-w$, and no two distinct vertices of $H$ may be placed in the same clique part, at least one vertex $w' \in H-w$ must be placed in an independent part $P'$. Since $w$ is not adjacent to $w'$ and both vertices belong to $H$, $w$ can be added to $P'$, resulting in an $M$-partition of $G$, and hence contradicting the minimality of $G$.

**Case 2.** If $P$ is a clique, then $H-w$ is a clique of size at least $k+1$, and so in the partition of $G-w$, at least one vertex of $H-w$ falls in a clique part $P'$. As in Case 1, $w$ can be added to $P'$, contradicting the minimality assumption. □

Since every matrix $M$ has finitely many split minimal obstructions, there is an obvious polynomial time algorithm for the $M$-partition problem. However, a more efficient algorithm can be obtained by using the method of 'sparse-dense' partitions [5]. (In our context, $k$-colourable graphs will be 'sparse', and $\ell$-co-colourable graphs, i.e., graphs whose complements are $\ell$-colourable, will be 'dense'.)

**Theorem 2.5.** *The $M$-partition problem restricted to split graphs can be solved in time $n^{O(k+\ell)}$.*

*Proof.* We may assume that $C$ has no asterisks, according to Fact 2.4. Further, we may also assume $k+\ell \geqslant 3$, otherwise we may use existing algorithms. It is shown in [5] that one can generate all partitions of $G$ into a $k$-colourable graph $G_A$, and an $\ell$-co-colourable graph $G_B$, in time $n^{2c}$, where $c$ is the maximum size of a subgraph of $G$ that is both $k$-colourable and $\ell$-co-colourable. Since $G$ is a split graph, it is easy to see that $c$ is at most $k+\ell$. We shall argue that for each such partition $G_A, G_B$ one can efficiently test (in time dominated by $n^{O(k+\ell)}$) whether an $M$-partition is possible. Thus in time $n^{O(k+\ell)}$ we can test all $(n^{O(k+\ell)})$ possible partitions $G_A, G_B$, and either find an $M$-partition of $G$, or conclude that none exists. Thus consider a fixed $G_A, G_B$ partitioning $G$, and consider also a split partition of $G$ into an independent set $I$ and a clique $K$. Clearly, $G_A$ has at most $k$ vertices in $K$, and $G_B$ has at most $\ell$ vertices in $I$. For these $k+\ell$ vertices we will consider all possible assignments into the $k+\ell$ parts of $M$. Since $k+\ell$ is fixed, we can just look at each one of such assignments separately. So we may assume that these $k+\ell$ vertices have been assigned, and focus on the remaining vertices. This means that now $G_A$ is an independent set and $G_B$ a clique. Thus, in any $M$-partition of $G$, the parts used by the vertices of $G_A$ form to a principal submatrix $A'$ of $A$ without 1's, and the parts used by the vertices of $G_B$ form to a principal submatrix $B'$ of $B$ without 0's. Since $M$ is fixed, we can examine each pair of such submatrices $A', B'$ separately. For a concrete $A', B'$ we can actually replace $A'$ by the all-zero matrix, and replace $B'$ by the all-one matrix, since $G_A$ is independent and $G_B$ is a clique. By our assumption, the submatrix $C'$ of $C$ corresponding to $A', B'$ has no asterisks, and so the resulting matrix $M'$ is asterisks-free. According to [5], such an $M'$-partition problem can be solved in time dominated by $n^{O(k+\ell)}$. □

By contrast, we note that there are matrices $M$ such that the list $M$-partition problem for split graphs is $NP$-complete. Indeed, in [4] it is proved that there are matrices $M$ for which the list $M$-partition problem for chordal graphs is $NP$-complete, and the graphs produced in that reduction happen to be split graphs.

## 3. A Special Class of Matrices

As seen in Section 2, for any $m \times m$ matrix $M$, there is an exponential upper bound on the size of a largest split minimal $M$-obstruction. In this section we show a family of matrices for which this bound is nearly tight.

For $k, t \in \mathbb{N}$, with $1 \leqslant t \leqslant k-1$, let $M_{k,t}$ be a $k \times k$ matrix with diagonal entries all zero, $t$ ones in row $k$, symmetrically, $t$ ones in column $k$ and asterisks everywhere else. By permuting the rows and columns of $M_{k,t}$

$$\begin{array}{ccc} P_1 & P_2 & P_3 \end{array}$$
$$\begin{pmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{pmatrix}$$
$$M_{3,1}$$

$$\begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array}$$
$$\begin{pmatrix} 0 & * & * & * & * \\ * & 0 & * & * & 1 \\ * & * & 0 & * & 1 \\ * & * & * & 0 & 1 \\ * & 1 & 1 & 1 & 0 \end{pmatrix}$$
$$M_{5,3}$$

$$\begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array}$$
$$\begin{pmatrix} 0 & * & * & * \\ * & 0 & * & * \\ * & * & 0 & 1 \\ * & * & 1 & 0 \end{pmatrix}$$
$$M_{4,1}$$

Figure 3.1: Matrices $M_{k,t}$ for $k \in \{3,4,5\}$ and $t \in \{1,3\}$

we assume without loss of generality that the one entries of row $k$ are in columns $k-t, ..., k-1$ and symmetrically, that the one entries of column $k$ are in rows $k-t, ..., k-1$. See Figure 3.1 for some examples.

**Theorem 3.1.** *Let $M = M_{k,t}$ where $k = 2t+1, t \in \mathbb{N}$. Then the size of the largest split minimal $M$-obstruction is $\Omega(2^k/\sqrt{k})$.*

*Proof.* Let $n \in nats, t = n, k = 2n+1$. An $M$-partition has $k = 2n+1$ parts. Let $P$ denote the part in row and column $2n+1$, and designate the $n$ parts that have a one to $P$ as *restricted parts*, $R_1, ..., R_n$ and the remaining $n$ parts as *unrestricted parts*, $U_1, ..., U_n$. See Figure 3.2.

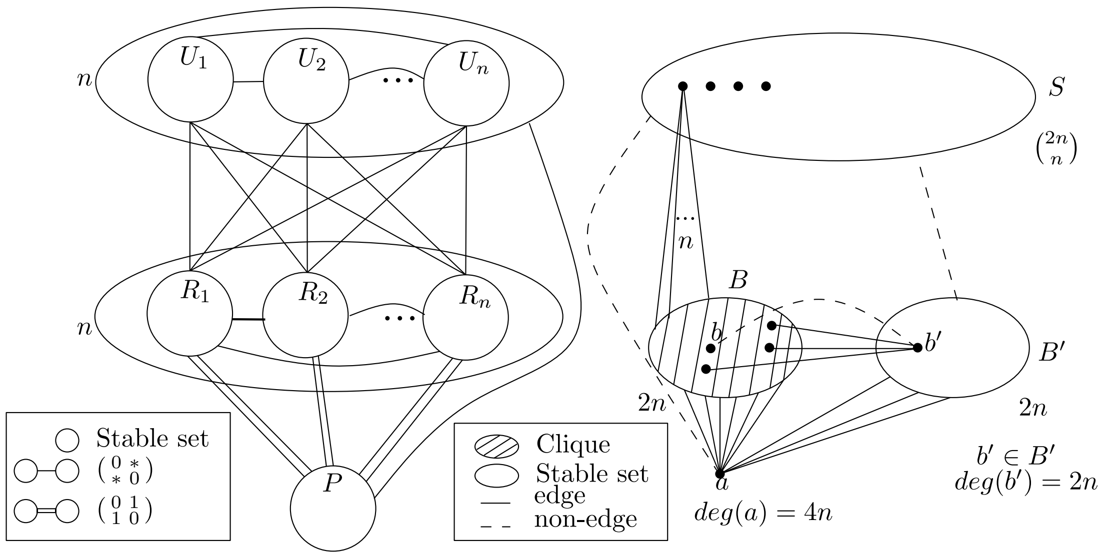


Figure 3.2: A partition corresponding to $M_{2n+1,n}$ (left), and an obstruction $G$ (right)

The minimal obstruction $G$, depicted in Figure 3.2, has a *special vertex* $a$, and $2n$ vertices forming a clique $B$, and are all adjacent to $a$ (so that $B \cup \{a\}$ is a clique of size $2n+1$). Further, $G$ has another $2n$ vertices forming an independent set $B'$ such that for each $b \in B$ there is a $b' \in B'$ that is not adjacent to $b$ but adjacent to every other vertex of $B \cup \{a\}$. Call $b$ and $b'$ *mates*. Finally, $G$ has an independent set $S$ of size $\binom{2n}{n}$ such that for every subset $\tilde{B}$ of $B$ of size $n$, there is exactly one vertex $s \in S$ adjacent to exactly the vertices of $\tilde{B}$. Note that $G$ is a split graph since $B \cup \{a\}$ is a clique and $B' \cup S$ is an independent set.

To see that $G$ is indeed an obstruction, suppose otherwise, and note that $B \cup \{a\}$ is a clique of size $2n+1$, so each of its vertices must be placed in a different part. Since each vertex of $B$ has a mate in $B'$ that is adjacent to $a$ and all of the other vertices in $B$, all parts other than the part containing $a$ have size at least two in any $M_{k,t}$-partition of $G$. Thus only the part containing $a$ may be a singleton. Further, the $2n+1$-st part $P$ must be the only singleton part, otherwise all of the restricted parts must be singletons (since $G$, being a split graph, contains no induxed $C_4$). Therefore $a \in P$. Now whichever $n$ vertices of $B$ are placed in the unrestricted parts, there is a vertex $s \in S$ adjacent to exactly these vertices, and so must be placed into one of the restricted parts. But as $s$ is not adjacent to $a$, it cannot be placed in a restricted part, and $s$ can't be added to $P$; hence, $G$ is not

$M_{k,t}$-partitionable.

To argue that $G$ is a minimal obstruction, we show that removing a vertex from one of $S, B, B'$, or $\{a\}$ allows us to $M$-partition the resulting graph:

(i) For $s \in S$ partition $G - s$ as follows: map $a$ to $P$, place each $b \in B$, together with its mate $b' \in B'$, in some part, making sure that neighbours of the missing $s$ are placed in unrestricted parts. Now each remaining vertex of $S$ has an unrestricted part to go to.

(ii) We consider $b \in B$ together with its mate $b' \in B'$. For $G - b$, place $a$ in $P$, place $b$'s mate $b'$ in an unrestricted part $P_{b'}$, and place all of $S$ and all of $B'$ in $P_{b'}$. This is possible since $B' \cup S$ form an independent set. Place the remaining $2n-1$ vertices of $B$ to the remaining $2n-1$ parts arbitrarily. To partition $G - b'$, place $b$ in $P$, and place $a$ together with all of the vertices of $S$ in an unrestricted part $P_a$, and place each other pair of mates $v, v'$ from $B$ and $B'$ into a distinct part, different from $P$ and $P_a$.

(iii) Finally, $G - a$ can be partitioned using the restricted and unrestricted parts only, not placing anything in $P$. Place each $b$ and its mate $b'$ into a part. Each $s \in S$ is only forbidden from $n$ out of the $2n$ parts and so it can always be placed somewhere.

Now $G$ has $\binom{2n}{n} + 4n + 1$ vertices, and using the fact that $\binom{2n}{n} = \Omega(\frac{4^n}{\sqrt{n}})$ we conclude that $G$ has $\Omega(\frac{2^k}{\sqrt{k}})$ vertices. In particular, $G$ is of size exponential in $k$. $\square$

## 4. Generalized Split Graphs

Split graphs can be viewed as a special case of $(k, \ell)$-graphs - those graphs whose vertices can be partitioned into $k$ independent sets and $\ell$ cliques. (Thus split graphs are the $(1, 1)$-graphs.)

In this section, we focus on $(k, \ell)$-graphs other than the $(1, 1)$-graphs. We begin with $(2, 0)$- and $(0, 2)$-graphs, and then discuss other $(k, \ell)$-graphs. Recall that the $(2, 0)$-graphs are the bipartite graphs, while the $(0, 2)$-graphs are the co-bipartite graphs. As it turns out, there are only finitely many bipartite and finitely many co-bipartite minimal obstructions, for any matrix $M$.

**Theorem 4.1.** *For any $m \times m$ matrix $M$, there are only finitely many bipartite minimal obstructions and only finitely many co-bipartite minimal obstructions.*

To prove Theorem 4.1 we use an approach similar in nature to that used Section 2. Starting with bipartite graphs, note that we may assume that the matrix $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$ is not a principal submatrix of the matrix $M$, otherwise the problem would be trivial.

**Proposition 4.2.** *Let $M$ be an $(A, B, C)$-block matrix, with $A$ of size $k \times k$ and $B$ of size $\ell \times \ell$. Suppose the block $A$ has no asterisk entries. If $G$ is an $M$-partitionable bipartite graph, then any part $P$ of $A$ in an $M$-partition of $G$ contains a homogeneous set of size at least $\frac{|P|}{2^{2\ell}}$*

*Proof.* Fix a bipartition of $G$ and let $P$ be a part of $A$ in an $M$-partition of $G$. As $A$ has no asterisks, the vertices of $P$ all have the same adjacency relation to vertices in other parts of $A$. Now let $P'$ be some part of $B$. Since $G$ is bipartite, $P'$ can have at most two vertices, one from each part of the bipartition of $G$. Let these vertices be $x$ and $y$. By the pigeonhole principle, $x$ is either adjacent to, or non-adjacent to, at least half of the vertices of $P$. Suppose with out loss of generality, that $x$ is adjacent to at least half of the vertices of $P$. Call these vertices $P_x$. Applying the pigeonhole principle again, this time to the vertex $y$, we have that $y$ is either adjacent to, or non-adjacent to, at least half of the vertices of $P_x$. Let the larger of these two sets be $P_{xy}$, and note that $P_{xy} \geqslant \frac{|P|}{2^2}$. Now there are $\ell - 1$ clique parts other than $P'$, each of size at most two. Inductively, we obtain a homogeneous set in $P$ of size at least $\frac{|P|}{2^{2\ell}}$ ◻

Theorem 4.1 now follows for bipartite graphs. The proof for co-bipartite graphs follows by complementation.

*Proof of Theorem 4.1.* As discussed above, we assume that $A$ contains no asterisk entries. We show that any bipartite minimal obstruction is of size at most

$$2^{2\ell}(k + \ell)(2\ell + 3)$$

Suppose otherwise, and let $G$ be a minimal obstruction with at least $2^{2\ell}(k + \ell)(2\ell + 3) + 1$ vertices. For an arbitrary vertex $v$, the graph $G - v$ is $M$-partitionable, and so some part $P$ in an $M$-partition of $G - v$ contains at least $2^{2\ell}(2\ell + 3)$ vertices. Since $2^{2\ell}(2\ell + 3) \geqslant 3$ for $l \geqslant 0$, and no clique part of $M$ may contain more than two vertices, $P$ must be an independent set. Thus by Proposition 4.2, $P$ contains a homogeneous set of $G - v$ of size at least $\frac{|P|}{2^{2\ell}} \geqslant 2\ell + 3$. By the pigeonhole principle, $G$ has an homogeneous set $H$ of size at least $\ell + 2$. Note that $H$ is an independent set. Let $h \in H$, and consider a partition of $G-h$. As there are only $\ell$ clique parts and $\ell+1$ vertices in $H-h$, there must be a part $P'$ of $A$ that contains a vertex $h'$ of $H-h$. But since $H$ is an independent set, and $h$ has the same neighbourhood as $h'$, we may add $h$ to $P'$ to obtain an $M$-partition of $G$, a contradiction. □

This completes a discussion of $(k,\ell)$ graphs with values of $k$ and $\ell$ that satisfy $k+\ell \leqslant 2$. We now consider $(k,\ell)$-graphs with $k+\ell \geqslant 3$. For convenience, let $(k,\ell)$ denote the set of $(k,\ell)$-graphs. The family of graphs depicted in Figure 1.1 is an infinite family of minimal obstructions to the matrix $M_{3,1}$[6]. We define the family more precisely as follows.

For $t \geqslant 3$, let $G(t)$ be the graph consisting of an even path on $2t$ vertices, and an additional vertex $u$, which is adjacent to all vertices of the path, except the endpoints.

**Theorem 4.3.** *If $k,\ell \in \mathbb{N}$ such that $k+\ell \geqslant 3$, then there exists a matrix $M$ that has infinitely many $(k,\ell)$-minimal obstructions.*

*Proof.* Note that for any $t \geqslant 3$, $G(t)$ is 3-colourable, and $G(t)$ is partitionable into a bipartite graph and a clique. That is, $G(t) \in (3,0) \cap (2,1)$. Therefore, for the matrix $M_{3,1}$, there are infinitely many minimal $(2,1) \cap (3,0)$ obstructions. By complementation, for any $t \geqslant 3$, the graph $\overline{G(t)}$ is in $(1,2) \cap (0,3)$, providing infinitely many $(1,2) \cap (0,3)$ obstructions for the matrix $\overline{M_{3,1}}$.

Now if $k \leqslant 1$, then since $k+\ell \geqslant 3$, it must be that $\ell \geqslant 2$, and so the family $\{\overline{G(t)} | t \geqslant 3\}$ is a family of $(k,\ell)$-minimal obstructions for $\overline{M_{3,1}}$. On the other hand, if $k \geqslant 2$, then the family $\{G(t) | t \geqslant 3\}$ is a family of $(k,\ell)$-minimal obstructions for the matrix $M_{3,1}$. □

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## References

[1] T. Feder and P. Hell. Matrix Partitions of Perfect Graphs. *Discrete Mathematics*, 306(19-20):2450–2460, October 2006.

[2] T. Feder and P. Hell. On Realizations of Point Determining Graphs, and Obstructions to Full Homomorphisms. *Discrete Mathematics*, 308(9):1639–1652, 2008.

[3] T. Feder, P Hell, and W. Hochstättler. Generalized Colourings (Matrix Partitions) of Cographs. In *Graph Theory in Paris*, pages 149–167. Birkhauser Verlag, 2006.

[4] T. Feder, P. Hell, S. Klein, L. T. Nogueira, and F. Protti. List Matrix Partitions of Chordal Graphs. *Theoretical Computer Science*, 349:52–66, 2005.

[5] T. Feder, P. Hell, Sulamita Klein, and Rajeev Motwani. List Partitions. *SIAM Journal on Discrete Mathematics*, 16(3):449, November 2003.

[6] T. Feder, P. Hell, and S. Rizi. Obstructions to Partitions of Chordal Graphs. *accepted in Discrete Mathematics*.

[7] T. Feder, P. Hell, and W. Xie. Matrix Partitions with Finitely Many Obstructions. *The Electronic Journal of Combinatorics*, 14, 2007.

[8] S. Földes and P.L. Hammer. Split Graphs. *Proc. 8th Southeastern Conf. on Combinatorics, Graph Theory and Computing (F. Hoffman et al., eds.), Louisiana State Univ., Baton Rouge, Louisiana. (As cited in Golumbic 2004)*, pages 311–315, 1977.

[9] P. Hell. Graph Partitions with Prescribed Patterns. *accepted in European Journal of Combinatorics*.

[10] D. König. Theorie der Endlichen und Unendlichen Graphen (as cited in West, 2001). 1936.

[11] P. Valadkhan. *Matrix Partitions of Special Graphs*. PhD thesis, Simon Fraser University, 2013.