On the Limit Average Degree of Iterated Line Graphs

Tomás Feder* Carlos Subi†

Abstract

Given a graph G, let $G^{(k)}$ denote the kth iterated line graph of G (i.e., $G^{(0)} = G$ and $G^{(k+1)}$ is the line graph of $G^{(k)}$). Let n_k and d_k denote the number of vertices and the average degree of $G^{(k)}$ respectively. We show that if G is not a path (the case of a path is trivial) then the sequence $2 + (d_k - 2)/2^k$ is nondecreasing and reaches a limit value d_{lim}. Since $n_{k+1} = n_k d_k/2$, it follows that $(\log_2 n_k)/k = (k-3)/2 + \log_2(d_{lim} - 2 + o(1))$. We conjecture that if G is obtained from H by adding one edge, then $n_k(H) = o(n_k(G))$, and we prove this conjecture for the case where H is regular.

1 Introduction

In most of this paper, a graph $G = (V, E)$ is a connected simple graph, without loops or parallel edges. Given a graph G, the line graph $L = L(G)$ is such that the vertices of L are the edges of G, and two distinct vertices of L share an edge if and only if the two corresponding edges of G share an endpoint. Given a graph G, we let $G^{(0)} = G$ and $G^{(k+1)} = L(G^{(k)})$ be an infinite sequence of graphs built from G.

Many conjectures have been proven or disproven about reconstructing trees [1, 2, 3, 4, 5, 6, 7, 8, 9]. We first examine the following known conjecture.

Conjecture 1 *Given two nonisomorphic trees G and H, there exists $k \geq 0$ such that $|V(G^{(k)})| \neq |V(H^{(k)})|$.*

We shall not prove this conjecture. We examine this conjecture in two special cases, namely *at-most-cubic trees*, with vertices of degrees 1,2, or 3, and *cubic trees*, with vertices of degrees 1 or 3. We show some general results that imply that cubic caterpillars satisfy the conjecture with regards to all other cubic trees.

We then show that the conjecture is false for any two nonisomorphic d-regular, n-vertex graphs (which are necessarily not trees).

We next show that $n_k = |V(G^{(k)})|$ goes to infinity for every graph G that is not a trivial graph. The trivial graphs are paths (with $n_k = \max(0, n_0 - k)$), cycles (with $n_k = n_0$) and a star with three leaves (with $n_0 = 4$ and $n_k = 3$ for $k \geq 1$). We conjecture the following.

Conjecture 2 *If G is not a trivial graph, and H is obtained from G by removing an edge, then $|V(H^{(k)})| = o(|V(G^{(k)})|)$.*

We prove special cases of this conjecture, including the case where H is regular. The approach is through a comparison of limit maximum and average degrees in H and G respectively.

We finally indicate that the results based on degree analysis extend to the case of multigraphs.

*268 Waverley Street, Palo Alto, CA 94301. Email: tomas@theory.stanford.edu

†carlos.subi@hotmail.com

2 Cubic and at-Most-Cubic Trees

Given an at-most-cubic tree G, we let $v_i(G)$ be the number of vertices of degree i in G for $i = 1, 2, 3$. We show the following.

Theorem 1 *Let G and H be two at-most-cubic trees with n vertices. Then $|V(G^{(k)})| = |V(H^{(k)})|$ for $k = 0, 1$, and $|V(G^{(2)})| = |V(H^{(2)})|$ if and only if $(v_1(G), v_2(G), v_3(G)) = (v_1(H), v_2(H), v_3(H))$.*

Proof. Let $v_i = v_i(G)$ and $v'_i = v_i(H)$. Since both graphs have the same number of vertices and edges, we have $v_1 + v_2 + v_3 = n = v'_1 + v'_2 + v'_3$ and $v_1 + 2v_2 + 3v_3 = 2n - 2 = v'_1 + 2v'_2 + 3v'_3$. From these we infer that $v_1 = v_3 + 2$, $v'_1 = v'_3 + 2$, and $v_2 + 2v_3 = n - 2 = v'_2 + 2v'_3$.

Since G and H have the same number of vertices and edges, the results follow for $k = 0, 1$. For $k = 2$, we observe that the number of edges in the graphs for $k = 1$ is $v_2 + 3v_3 = n - 2 + v_3$ and $v'_2 + 3v'_3 = n - 2 + v'_3$. Therefore these two agree if and only if $v_3 = v'_3$, and this condition is equivalent to $v_i = v'_i$ for $i = 1, 2, 3$. □

From this result, we infer the following.

Theorem 2 *Let G and H be two cubic trees with n vertices. Then $|V(G^{(k)})| = |V(H^{(k)})|$ for $k = 0, 1, 2, 3$, and $|V(G^{(4)})| = |V(H^{(4)})|$ if and only if $(v_1(\hat{G}), v_2(\hat{G}), v_3(\hat{G})) = (v_1(\hat{H}), v_2(\hat{H}), v_3(\hat{H}))$ for the trees $\hat{G}, \hat{H}$ obtained from G, H by removing all leaves.*

Proof. The graphs $G^{(1)}, H^{(1)}$ have one triangle for each vertex of $\hat{G}, \hat{H}$ joined at vertices. The graphs $G^{(2)}, H^{(2)}$ have one triangle for each vertex of $\hat{G}, \hat{H}$ and a complete bipartite $K_{2,2}$ joining triangles for each edge of $\hat{G}, \hat{H}$. Therefore $|V(G^{(k)}| = |V(H^{(k)})|$ for $k = 0, 1, 2, 3$. For the edges of $G^{(3)}, H^{(3)}$, obtained from adjacent edges of $G^{(2)}, H^{(2)}$, that are in the same number for edges (1) within a triangle, (2) within a $K_{2,2}$, or (3) joining a triangle with a $K_{2,2}$, by counting vertices and edges of $\hat{G}, \hat{H}$. Indeed, these are 3 times the number of vertices for (1), 4 times the number of edges for (2), and 16 times the number of edges for (3). However, for edges joining two $K_{2,2}$, these are four times the number of vertices of $\hat{G}^{(2)}, \hat{H}^{(2)}$ and the result follows from Theorem 1. □

The result for caterpillars follows.

Corollary 1 *If G is a cubic caterpillar and H is a cubic tree, both with n vertices, and with H not isomorphic to G, then $|V(G^{(4)})| \neq |V(H^{(4)})|$.*

Proof. After removing leaves, caterpillars are the only trees that become paths that have $v_1 = 2$. □

We show that Conjecture 1 contradicts the following conjecture.

Conjecture 3 *[7] There exists $k > 0$ such that for any two graphs G and H that are not trivial, $|V(G^{i})| = |V(H^{(i)}|)$ for all $0 \le i \le k$ implies $|V(G^{j})| = |V(H^{(j)}|)$ for all j.*

Theorem 3 *For any k, there exists two nonisomorphic trees G and H such that $|V(G^{i})| = |V(H^{(i)}|)$ for all $0 \le i \le k$.*

Proof. The number of nonisomorphic trees of size n is exponential in n, while the number of such sequences $|V(G^{i})|$ is only polynomial in n for fixed k, since $|V(G^{i})| \le n^{2^i}$. □

3 Regular Graphs and Other Graphs

The following results disproves the first conjecture for regular graphs (which are not trees though).

Theorem 4 *[7] If G is an n-vertex d-regular graph, $d \geq 3$, then $G^{(k)}$ is an n_k-vertex d_k-regular graph, where*

$$\begin{aligned} n_k(n,d) &= n2^{k(k-3)/2}((d-2)+2)((d-2)+1)\cdots((d-2)+2^{2-k}) \\ &= \Theta(n2^{k(k-3)/2}(d-2)^k) \end{aligned}$$

and $d_k(n,d) = 2 + 2^k(d-2)$.

Proof. Notice that if G is an n-vertex d-regular graph, then $G^{(k)}$ is an n_k-vertex d_k-regular graph, where $n_0 = n$, $d_0 = d$, $d_{k+1} = 2(d_k - 1)$ and $n_{k+1} = n_k d_k/2$.

Furthermore, $d_{k+1} - 2 = 2(d_k - 2)$, so $d_k - 2 = 2^k(d-2)$. Also,

$$\begin{aligned} n_k &= n_0(d_0/2)\cdots(d_{k-1}/2) \\ &= (n/2^k)(2+(d-2))(2+2(d-2))\cdots(2+2^{k-1}(d-2)) \\ &= n2^{1+2+\cdots+(k-1)-k}((d-2)+2)((d-2)+1)\cdots((d-2)+2^{2-k}) \\ &= n2^{k(k-3)/2}((d-2)+2)((d-2)+1)\cdots((d-2)+2^{2-k}) \\ &= \Theta(n2^{k(k-3)/2}(d-2)^k). \end{aligned}$$

□

It is easy to see that if G is a graph of minimum and maximum degrees d_{min} and d_{max} respectively, then $n_k(n, d_{min})$ and $n_k(n, d_{max})$ are lower and upper bounds on the number of vertices of $G^{(k)}$ respectively [7], while $n_{k-1}(d, d-1)$ is a lower bound for $d = d_{max} \geq 4$ since $G^{(1)}$ contains a clique of size d. If $d = 3$ but G is not trivial, then $G^{(4)}$ contains a clique of size 4, so $n_{k-4}(4,3)$ is a lower bound.

For the second conjecture of the introduction, notice that if G is a star with r leaves, then H is a star with $r-1$ leaves, so $G^{(1)}, H^{(1)}$ are cliques of sizes r ane $r-1$, and the conjecture follows from the theorem. If G is a clique of size 4, then $G^{(2)}$ has all degrees 6, while $H^{(2)}$ has maximum degree 5, and the conjecture follows from the theorem as well. We do not know what happens if G is a clique of size 5.

Let n_k and d_k denote the number of vertices and average degree of $G^{(k)}$.

Theorem 5 *For all $k \geq 0$, $d_{k+1} \geq 2(d_k - 1)$, with equality if only if $G^{(k)}$ is reqular. Therefore if $d_0 > 2$, then $n_k \geq n_k(n_0, d_0)$.*

Proof. By convexity,

$$\begin{aligned} n_k d_k(d_k - 1)/2 &\leq \sum_{v \in V(G^k)} d(v)(d(v)-1)/2 \\ &= n_{k+2} = n_{k+1}d_{k+1}/2 = n_k d_k d_{k+1}/4, \end{aligned}$$

with equality only if all $d(v)$ are equal, so the first claim follows, and we have $d_(k+1) \geq 2(d_k - 1)$, so $d_i \geq 2 + 2^i(d_0 - 2)$, and the second claim follows as in the Theorem 4,

$$n_k \geq n2^{k(k-3)/2}((d_0-2)+2)((d_0-2)+1)\cdots((d_0-2)+2^{2-k}),$$

so $n_k \geq n_k(n_0, d_0)$. □

Corollary 2 *Suppose $H^{(k)}$ has maximum degree $d^{(k)}$ and $G^{(k)}$ has average degree d_k. If $d^{(k)} < d_k$ for some $k = \hat{k}$, then $n_k(H) = o(n_k(G))$ as $k \to \infty$.*

Proof. We hat $n_k(H)/n_k(G) = \Theta(((d^{(\hat{k})} - 2)/((d_{\hat{k}})^k - 2)) = o(1)$. □

For the second conjecture of the introduction, we prove two results.

Corollary 3 *If H is d-regular and G is obtained from H by adding one edge e (possibly joining to a new vertex v) then $n_k(H) = o(n_k(G))$.*

Proof. If the new edge e in G joins two vertices already in H, then the maximum degree in H is $d^{(0)} = d$ while the average degree of G is d_0 and satisfies $d^{(0)} < d_0$.

If e joins a new vertex v to the graph H to obtain G, then $d^{(1)} = 2(d-1)$ while $G^{(1)}$ has a vertex for e of degree d and d vertices of degree $2d-1$, while the remaining vertices have degrees $2d-2$ as for $H^{(1)}$, so $d^{(1)} < d_1$. □

If H is not regular but $L(H)$ is regular, then there exist positive integers $a > b$ such that H is a bipartite graph with degrees a in one side and degrees b in the other side of the bipartition. We call such an H an almost regular bipartite graph. Note that if $H = L(K)$ for some graph K in turn, then all degrees must be at most two in K, otherwise H would contain a triangle an would not be bipartite, so K is trivial.

Corollary 4 *If H is an almost regular bipartite graph and G is obtained from H by adding one edge e (possibly joining to a new vertex v), then $n_k(H) = o(n_k(G))$.*

Proof. Since $H^{(1)}$ is regular and $G^{(1)}$ is obtained from $H^{(1)}$ by adding one vertex corresponding to e, it follows from Corollary 3 that $d^{(1)} < d_1$ or $d^{(2)} < d_2$. □

Other than in the two cases where $G^{(k)}$ becomes regular, the sequence $2 + (d_k - 2)/2^k$ is stricly increasing by Theorem 5, and tends to a limit limit $d_{\lim}$ since the corresponding sequence for the maximum degree $2 + (d^{(k)} - 2)/2^k$ is nonincreasing and upper bounds it.

Theorem 6 *If $d_{\lim} > 2$, then $n_k = n_k(n_0, d_{\lim} + o(1))$.*

Proof. The proof follows as in Theorems 4 and 5. Since $d_i \leq 2 + 2^i(d_{\lim} - 2)$, we have

$$n_k \leq n2^{k(k-3)/2}((d_{\lim} - 2) + 2)((d_{\lim} - 2) + 1) \cdots ((d_{\lim} - 2) + 2^{2-k}),$$

and so $n_k \leq n_k(n, d_{\lim})$. On the other hand, since for arbitrarily small $\epsilon > 0$, all but a constant number j of factors in the product upper bounded with $d_{\lim}$ give at least the value with $d_{\lim} - \epsilon$, we have for large enough k that $n_k \geq n_k(n, d_{\lim} - 2\epsilon)$, and the result follows. □

It has been shown [5] that for all graphs G that are not paths, there exists $K \geq 0$ such that for all $k \geq K$, then if $G^{(k)}$ has maximum degree $d^{(k)}$, then $d^{(k+1)} = 2d^{(k)} - 2$.

It follows from Theorem 4 for regular graphs that the diameter of $G^{(k)}$ for G regular and not trival is $\Omega(k)$, by degree versus number of vertices expansion analysis, while an upper bound for it of $D + k$ if G has diameter D follows from the fact that line graphs can only increase diameter by at most one. The lower bound can be improved to $k - n_3 - 3$ since $D_{k+1} = D_k + 1$ for $k \geq 3 + n_3$ as shown in [7].

4 Multigraphs

In this section, we consider the case where G is a multigraph, possibly with loops and parallel edges. We define the line graph $L(G)$ of a multigraph G in such a way as to preserve the property that if an edge $e = (x, y)$ in G joins two vertices x of degree d_1 and y of degree d_2, then the degree of e in $L(G)$ is $d_1 + d_2 - 2$ as for the special case where G is a graph. In particular, if $d_1 = d_2 = d$, then this degree is $2d - 2$.

Notice that a loop at a vertex v adds degree 2 to the degree of v. If two edges e_1 and e_2 in G share an endpoint, then they are joined in $L(G)$ by a single edge if e_1 and e_2 are not loops and are not parallel to each other in G, but are joined in $L(G)$ by two parallel edges if e_1 and e_2 are parallel to each other in G, of if one of them is a loop and the other one is not. Finally, if e_1 and e_2 are both loops at the same vertex v of G, then they are joined by four parallel edges in $L(G)$, and if e is a loop in G then we add a loop for e in $L(G)$.

It is easy to verify that this definition of line graph for a multigraph satisfies the degree property claimed, and that results from the previous section then extend to multigraphs. Finally, if we change this definition of line graphs for multigraphs by saying that each edge in G is adjacent to itself as well, then we include an additional loop at each vertex of $L(G)$, and the degree equations get simplified from $d_1 + d_2 - 2$ and $2d - 2$ to $d_1 + d_2$ and $2d$. In this case $n_k(n, d)$ gets transformed to $n'_k(n, d) = n2^{k(k-3)/2}d^k$ and similar results follow.

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