A Dichotomy Theorem on Fixed Points of Several Nonexpansive Mappings

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Abstract

The problem of finding a fixed point of a nonexpansive mapping on a hypercube has a polynomial time algorithm. In fact, it is known that one can find a 2-satisfiability characterization of the set of all fixed points in polynomial time. This implies that the problem of finding a vertex that is a common fixed point of several given nonexpansive mappings on a hypercube has a polynomial time algorithm.

We consider the problem of finding a vertex that is a common fixed point of several given nonexpansive mappings on a more general cartesian product of graphs. For a single nonexpansive mapping, a known polynomial time algorithm finds a fixed point and a 2-satisfiability-like characterization of all fixed points. We introduce graphs with a farthest point property (also called *apiculate graphs* in [1]), and show that finding a common fixed point of several nonexpansive mappings on cartesian products of such graphs has a polynomial time algorithm. We generalize this result to any family of graphs having a majority function.

By contrast, the smallest graph (in the sense of having the fewest vertices, and the fewest edges out of those having the fewest vertices) without the fathest point property is $K_{2,3}$, and finding a vertex that is a fixed point of two given nonexpansive mappings (retractions) on a cartesian product of graphs isomorphic to $K_{2,3}$ is NP-complete. More generally, we exhibit an infinite family of graphs without the farthest point property giving NP-completeness. We show that for any family of graphs not having a majority function, the existence of a common fixed point of two nonexpansive mappings on cartesian products of such graphs is NP-complete. This proves a dichotomy for the problem based on the existence of a majority function; a similar dichotomy is obtained for the special case of nonexpansive mappings that are retractions. We finally characterize the families of chordal graphs corresponding to both dichotomies.

1 Introduction

Many results in computational complexity take the form of a dichotomy theorem, where every problem in a given class is shown to be either polynomial time solvable or NP-complete. An early result is Schaefer's dichotomy of boolean constraint satisfaction problems [7]. Beyond the boolean domain, an approach to general constraint satisfaction was proposed by Feder and Vardi [5]. Recently, Bulatov has classified 3-element and conservative constraint satisfaction problems, with two dichotomy theorems [2, 3].

A similar dichotomy and classification project, for network stability problems, was initiated by Mayr and Subramanian [8, 6, 9]. They showed that every boolean network stability problem is either monotone, linear, adjacency-preserving, or NP-complete. They also showed that the monotone and linear cases are polynomial, and that a family of adjacency-preserving problems, the scatter-free case (containing stable matching as a special case) is polynomial as well. The general adjacency-preserving case was studied as the problem of finding a fixed point of a nonexpansive mapping on a hypercube, and shown polynomial by Feder [4], also generalizing the nonexpansive case to other cartesian products of graphs, and thus extending the results beyond the boolean domain.

The n-dimensional hypercube (or n-cube) is the graph $G = (V, E)$, where V consists of all n-bit vectors $x = x_1x_2 \cdots x_n$, $x_i \in \{0, 1\}$, and two vertices x, y in V are joined by an edge in E if there exists an $1 \leq i \leq n$ such that $x_i \neq y_i$, and $x_j = y_j$ for all $j \neq i$. The distance $d(x, y)$ between two vertices x, y in the n-cube equals the number of positions $1 \leq i \leq n$ such that $x_i \neq y_i$.

A mapping $f : V(G) \to V(G)$ on the n-cube G is *nonexpansive* if $d(f(x), f(y)) \leq d(x, y)$ for all vertices x, y in G. A *fixed point* of a nonexpansive mapping f is a vertex x such that $f(x) = x$. Assuming that a nonexpansive f on the n-cube is given by a black box that can be queried in polynomial time, that is, we specify an input x to f and the black box gives the image $f(x)$, Feder [4] gave a polynomial time algorithm for finding a fixed point x of a nonexpansive f on the n-cube, if one exists, and a second polynomial time algorithm that finds a 2-satisfiability instance on the variables x_i whose set of solutions characterizes the set of fixed points $x = x_1x_2 \cdots x_n$ of the nonexpansive mapping f.

Suppose we are given a collection of nonexpansive mappings $f_i : V(G) \to V(G)$ on the n-cube G, $1 \leq i \leq m$, and wish to find a common fixed point x with $f_i(x) = x$ for all i. We may then combine the m corresponding 2-satisfiability instances to obtain a single 2-satisfiability instance characterizing the set of all common fixed points, which can be solved in polynomial time to find a common fixed point, if one exists.

We study a generalization of the problem of finding common fixed points of a collection of nonexpansive mappings to cartesian products of graphs, which include the hypercube as the simplest special case. Given n graphs $G_1, G_2, \ldots, G_n$, their *cartesian product* $G = G_1 \Box G_2 \Box \cdots \Box G_n$ has vertices $V(G)$ given by $x = x_1 x_2 \cdots x_n$ with $x_i \in V(G_i)$, and two vertices x, y in $V(G)$ are joined by an edge in $E(G)$ if there exists an $1 \leq i \leq n$ such that (x_i, y_i) is an edge in $E(G_i)$, and $x_j = y_j$ for all $j \neq i$. The distance function d on G satisfies $d(x,y) = \sum_{1 \leq i \leq n} d_i(x_i, y_i)$, where d_i is the distance function on G_i.

A mapping $f : V(G) \to V(G)$ on a cartesian product $G = G_1 \Box G_2 \Box \cdots \Box G_n$ is *nonexpansive* if $d(f(x), f(y)) \leq d(x, y)$ for all vertices x, y in $V(G)$. A *fixed point* of a nonexpansive mapping f on G is a vertex x such that $f(x) = x$. Assuming again that f is given by a black box that can be queried in polynomial time, Feder [4] showed the following.

Theorem 1.1 *There is a polynomial time algorithm (polynomial in the sum of the sizes of the G_i) that finds sets $S_{ij} \subseteq V(G_i \Box G_j)$ for all $1 \leq i < j \leq n$, such that given a partial assignment of values $a_i \in V(G_i)$ for $i \in S \subseteq \{1, 2, \ldots, n\}$ with $|S| \geq 2$, there exists a fixed point x of the nonexpansive mapping f such that $x_i = a_i$ for all $i \in S$ if and only if $a_i a_j \in S_{ij}$ for all $1 \leq i < j \leq n$ with $i, j \in S$. The partial assignment of values a_i can thus be extended to a fixed point x in polynomial time by considering the sets S_{ij}.*

Suppose now we are given a collection of nonexpansive mappings $f_i : V(G) \to V(G)$ on a cartesian product $G = G_1 \Box G_2 \Box \cdots \Box G_n$, with $1 \leq i \leq m$. We wish to determine whether the question of the existence of a common fixed point x with $f_i(x) = x$ for all $1 \leq i \leq m$ can be solved in polynomial time. We obtain both positive and negative answers to this question.

Given vertices x_i, y_i in $V(G_i)$, the *interval* $I(x_i, y_i)$ is the set of vertices t_i in G_i such that $d(x_i, y_i) = d(x_i, t_i) + d(t_i, y_i)$. We say that G_i satisfies the *farthest point* property if for all vertices x_i, y_i, z_i there is a unique vertex t_i in $I(x_i, y_i) \cap I(x_i, z_i)$ that maximizes $d(x_i, t_i)$ over all t_i in $I(x_i, y_i) \cap I(x_i, z_i)$. Note that in particular cliques and cycles satisfy the farthest point property, and that the cartesian products and the retracts of graphs satisfying the farthest point property also satisfies the farthest point property.

We give a polynomial time algorithm for finding a common fixed point x with $f_i(x) = x$ for all $1 \leq i \leq m$ in the cartesian product $G = G_1 \Box G_2 \Box \cdots \Box G_n$, when all G_i satisfy the farthest point property. We generalize this result to any family $\mathcal{G}$ of graphs G_i having a majority function. The results are shown in a more general form analogous to Theorem 1.1 involving a structural property that we show only holds for families $\mathcal{G}$ of graphs G_i having a majority function.

The smallest graph not satisfying the farthest point property is the complete bipartite graph $K_{2,3}$. We define a family of graphs $H_{a,b,c}$ for integers $a, b, c \geq 1$ that do not satisfy the farthest point property; in particular $a = b = c = 1$ gives $H_{1,1,1} = K_{2,3}$. We consider the cartesian product $G = G_1 \Box G_2 \Box \cdots \Box G_n$, in the case where all G_i are isomorphic to a given $H_{a,b,c}$. We then define pairs of nonexpansive mappings $f_1, f_2 : V(G) \to V(G)$ that are *retractions*, i.e., these mappings satisfy $f_i(f_i(x)) = f_i(x)$ for all vertices x in G. We show that for all choices of $H_{a,b,c}$ with $a, b, c \geq 1$, the question of whether two such mappings have a common fixed point $f_1(x) = f_2(x) = x$ is NP-complete.

We show that if a family $\mathcal{G}$ of graphs G_i does not have a majority function, then the existence of a common fixed point of two nonexpansive mappings f_i on cartesian products of graphs in $\mathcal{G}$ is NP-complete. This proves the dichotomy and classification of the problem of finding common fixed points of several nonexpansive mappings on cartesian products of graphs in $\mathcal{G}$, as polynomial or NP-complete depending on whether $\mathcal{G}$ has a majority function or not. A similar dichotomy holds for the special case of retraction mappings.

We finally characterize the families of chordal graphs that are polynomial or NP-complete for the problems of fixed points of multiple nonexpansive mappings and of fixed points of multiple retraction mappings.

2 Common fixed points with farthest point property and with majority function

Let $f_i : V(G) \to V(G)$ be nonexpansive mappings on a cartesian product $G = G_1 \Box G_2 \Box \cdots \Box G_n$, with $1 \leq i \leq m$. Assume again that f_i is given by a black box that can be queried in polynomial time. We generalize Theorem 1.1 in the case where each G_i satisfies the farthest point property.

Theorem 2.1 *Suppose each G_i satisfies the farthest point property. There is a polynomial time algorithm that finds sets $S_{ij} \subseteq V(G_i \Box G_j)$ for all $1 \leq i < j \leq n$, such that given a partial assignment of values $a_i \in V(G_i)$ for $i \in S \subseteq \{1, 2, \ldots, n\}$ with $|S| \geq 2$, there exists a common fixed point x of the nonexpansive mappings f_i such that $x_i = a_i$ for all $i \in S$ if and only if*

$a_i a_j \in S_{ij}$ for all $1 \le i < j \le n$ with $i, j \in S$. The partial assignment of values a_i can thus be extended to a common fixed point x with $f_i(x) = x$ for all $1 \le i \le m$ in polynomial time by considering the sets S_{ij}.

Proof. We can apply Theorem 1.1 to find the corresponding sets S_{ij}^k for each f_k. We can then define sets T_{ij} that are the intersection of all S_{ij}^k over all $1 \le k \le m$. Define a mapping $g_i(x_i, y_i, z_i) = t_i$ on $V(G_i)$ to be the unique t_i maximizing $d(x_i, t_i)$ for $t_i \in I(x_i, y_i) \cap I(x_i, z_i)$, by the definition of the farthest point property. Note that if x, y, z are fixed points of f_k, so that $f_k(x) = x, f_k(y) = y, f_k(z) = z$, then $t = t_1 t_2 \cdots t_n$ with $t_i = g_i(x_i, y_i, z_i)$ satisfies $t \in I(x, y) \cap I(x, z)$, $f_k(t) \in I(x, y) \cap I(x, z)$, and $d(x, t) = d(x, f_k(t))$, so $f_k(t)_i \in I(x_i, y_i) \cap I(x_i, z_i)$, and $d(x_i, t_i) = d(x_i, f_k(t)_i)$, therefore $f_k(t)_i = t_i$ and $f_k(t) = t$. That is, the vertex t is also a fixed point of f_k.

Thus the sets S_{ij}^k are such that if $x_i x_j, y_i y_j, z_i z_j \in S_{ij}^k$ then $g_i(x_i, y_i, z_i) g_i(x_j, y_j, z_j) \in S_{ij}^k$. Furthermore, we have $g_i(x_i, x_i, y_i) = g_i(x_i, y_i, x_i) = g_i(y_i, x_i, x_i) = x_i$. Such a function g_i is called a majority function. We may assume that the graphs G_i are disjoint, and extend the mappings g_i on each G_i to a single mapping g on $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$ satisfying $g(x, x, y) = g(x, y, x) = g(y, x, x) = x$. Furthermore, the sets T_{ij} are closed under g, that is, if $x_i x_j, y_i y_j, z_i z_j \in T_{ij}$ then $g(x_i, y_i, z_i) g(x_j, y_j, z_j) \in T_{ij}$. The mapping g is thus a *majority function* for the sets T_{ij}. The problem of simultaneously satisfying the conditions $x_i x_j \in T_{ij}$ when there exists a majority function is a constraint satisfaction that can be solved in polynomial time, as shown by Feder and Vardi [5]. In fact the algorithm in [5] infers smaller sets $S_{ij} \subseteq T_{ij}$ from the sets T_{ij}, by enforcing the condition that for all $l \neq i, j$, if $x_i x_j \in S_{ij}$, then there exists an x_l such that $x_i x_l \in S_{il}$ and $x_j x_l \in S_{jl}$ (otherwise $x_i x_j$ is removed from S_{ij}). As shown in [5], these inferred sets S_{ij} satisfy the condition that any partial solution can be extended to a full solution, as in the statement of the Theorem, completing the proof. $\square$

A family of graphs $\mathcal{G}$ has a *majority function* if each graph G_i in $\mathcal{G}$ has a function g_i such that $g_i(x_i, x_i, y_i) = g_i(x_i, y_i, x_i) = g_i(y_i, x_i, x_i) = x_i$ for all x_i, y_i in $V(G_i)$, and for every pair of graphs G_i, G_j in $\mathcal{G}$, if f is a nonexpansive mapping on $G_i \square G_j$, and $x_i x_j, y_i y_j, z_i z_j$ are fixed points of f, then $g_i(x_i, y_i, z_i) g_j(x_j, y_j, z_j)$ is a fixed point of f as well.

Note from Theorem 2.1 that the family of graphs with the farthest point property has a majority function. Another example of graphs with a majority function is the family of graphs G_i such that for all x_i, y_i, z_i in $V(G_i)$ there is a unique vertex t_i in $V(G_i)$ minimizing $d(x_i, t_i) + d(y_i, t_i) + d(z_i, t_i)$.

Theorem 2.2 *The statement of Theorem 2.1 holds for a family $\mathcal{G}$ of graphs G_i if and only if $\mathcal{G}$ has a majority function.*

Proof. Suppose $\mathcal{G}$ has a majority function. Obtain the sets S_{ij}^k as in Theorem 2.1. The nonexpansive mapping f_k on G can be projected to a nonexpansive mapping $f_{k,ij}$ on $G_i \square G_j$ whose fixed points are given by S_{ij}^k as shown in [4]. The projection $f_{k,ij}$ is obtained by fixing inputs x_i, x_j, finding a periodic point p for the resulting function on the remaining positions other than i, j, and obtaining the outputs y_i, y_j, independently of the choice of periodic point p. Thus the S_{ij}^k are closed under the majority functions g_i, g_j, and the result follows as in Theorem 2.1.

If $\mathcal{G}$ does not have a majority function, then by the Compactness Theorem, the family $\mathcal{G}$ has a finite subfamily $\mathcal{G}'$ that does not have a majority function. Then the sets S_{ij}^k that can be defined as fixed points of nonexpansive mappings $f_{k,ij}$ do not satisfy the 2-*Helly property*, since the 2-Helly property is equivalent to the existence of a majority function, as shown by Feder and Vardi [5]. The 2-Helly property is the statement that a partial assignment of a_i can be extended to a full solution if and only if partial assignments consisting of just pairs $a_i a_j$ can be extended to a full solution, as in the statement of Theorem 2.1. $\square$

3 Common fixed points without farthest point property and without majority function

Given integers $a, b, c \geq 1$, let $H_{a,b,c}$ be the graph consisting of five particular vertices x, y, z, u, v, and six paths joining them, with two paths of length a, from x to u and from x to v, two paths of length b, from y to u and from y to v, and two paths of length c, from z to u and from z to v. Note that in particular, for $a = b = c = 1$, the graph $H_{1,1,1}$ is a complete bipartite graph $K_{2,3}$ with u, v in one side and x, y, z in the other. The graphs $H_{a,b,c}$ do not satisfy the farthest point property, since $d(x,t)$ is maximized for $t \in I(x,y) \cap I(x,z)$ at both $t = u$ and $t = v$.

For a particular choice of integers $a, b, c \geq 1$, consider the cartesian product $G = G_1 \square G_2 \square \cdots \square G_n$, where each G_i is isomorphic to $H_{a,b,c}$, so that vertices x, y, z, u, v in $H_{a,b,c}$ correspond to vertices x_i, y_i, z_i, u_i, v_i in G_i.

A retraction on G is a nonexpansive mapping $f : V(G) \to V(G)$ such that $f(f(x)) = f(x)$ for all $x \in V(G)$. We define several retractions on G. The retractions f_{u_i} map r to s in such a way that $s_j = r_j$ for $j \neq i$,

$s_i = r_i$ for r_i on the paths joining u_i to x_i, y_i, z_i respectively, and if r_i is on a path joining v_i to one of x_i, y_i, z_i, then s_i is the corresponding vertex on the corresponding path joining u_i to one of x_i, y_i, z_i respectively. Similarly, the retractions f_{v_i} map r to s in such a way that $s_j = r_j$ for $j \neq i$, $s_i = r_i$ for r_i on the paths joining v_i to x_i, y_i, z_i respectively, and if r_i is on a path joining u_i to one of x_i, y_i, z_i, then s_i is the corresponding vertex on the corresponding path joining v_i to one of x_i, y_i, z_i respectively.

For $t_i \in \{x_i, y_i, z_i\}$ and $w_j \in \{x_j, y_j, z_j\}$, the retractions f_{t_i,w_j} compose f_{u_i} with f_{u_j}, and then with the mapping taking r to s in such a way that $s_k = r_k$ for $k \neq i, j$, and $s_i = r_i, s_j = r_j$ unless $r_i = t_i$ and $r_j = w_j$, in which case s_i and s_j are the neighbors of t_i and w_j on paths to u_i and u_j respectively.

Theorem 3.1 *For each choice of integers $a, b, c \geq 1$, one can define retractions f_1, f_2 as compositions of the retractions $f_{u_i}, f_{v_i}, f_{t_i,w_j}$ such that the existence of a common fixed point $f_1(x) = f_2(x) = x$ is NP-complete.*

Proof. An instance of the one-in-three satisfiability problem has a collection of variables, and triples of variables (X, Y, Z), and asks whether there exists an assignment of values in $\{0, 1\}$ to each variable in such a way that for each triple (X, Y, Z) in the instance, one of X, Y, Z has value 1 and the other two have value 0. The one-in-three satisfiability problem is NP-complete [7].

Let n be the number of triples (X, Y, Z) in a one-in-three satisfiability problem, so that each G_i corresponds to one such triple, with a correspondence between X, Y, Z and x_i, y_i, z_i in G_i. Let f_1 be the composition of the retractions f_{v_i} for $1 \leq i \leq n$. Let f_2 be the composition of the retractions f_{u_i} for $1 \leq i \leq n$, composed with the retractions f_{t_i,w_j} such that t_i and w_j correspond to two distinct variables T and W in a triple (X, Y, Z) in the one-in-three satisfiability instance.

The mappings f_1 and f_2 are retractions. Since the range of f_1 contains only vertices s such that s_i is on a path from v_i to one of x_i, y_i, z_i, and the range of f_2 contains only vertices s such that s_i is on a path from u_i to one of x_i, y_i, z_i, it follows that the only candidate fixed points $f_1(s) = f_2(s) = s$ have all $s_i \in \{x_i, y_i, z_i\}$. A choice of such an s with $s_i \in \{x_i, y_i, z_i\}$ thus chooses a variable from the corresponding triple (X, Y, Z). Furthermore, if X is chosen for one such triple, then X will also be chosen for all other triples containing X, otherwise $f_2(s) \neq s$, since the retractions f_{t_i,w_j} used in the definition of f_2 guarantee that we cannot choose two distinct variables T and W that share a triple. Thus the choice of s_i satisfies $f_1(s) = f_2(s) = s$ if and only if the s_i choose an element from $\{x_i, y_i, z_i\}$ corresponding to a solution to the one-in-three satisfiability problem. $\square$

We further show the following.

Theorem 3.2 *Let $\mathcal{G}$ be a family of graphs that does not have a majority function. Then the question of whether several nonexpansive mappings f_i on a cartesian product G of graphs G_i in $\mathcal{G}$ have a common x such that $f_i(x) = x$ for all f_i is NP-complete.*

Proof. As in the proof of Theorem 2.2, if $\mathcal{G}$ does not have a majority function, then $\mathcal{G}$ has a finite subfamily $\mathcal{G}'$ that does not have a majority function, and then the sets S_{ij}^k that can be defined as fixed points of nonexpansive mappings $f_{k,ij}$ do not satisfy the 2-Helly property. This means that one can define an instance of the problem with variables x_i constrained by sets S_{ij}^k of fixed points of $f_{k,ij}$ in such a way that for some $r \geq 3$, there exist values $a_1, a_2, \ldots, a_r$ such that no solution x has $x_i = a_i$ for all $1 \leq i \leq r$, but for each $1 \leq i \leq r$, there exists a solution x^i that has $x_j^i = a_j$ for all $1 \leq j \leq r$ with $j \neq i$.

Let b_i be a possible value for x_i^i which is closest to a_i in G_i. Let P_i denote a shortest path from a_i to b_i in G_i. We may restrict each variable x_i with $1 \leq i \leq r$ ranging over G_i to the path P_i, by considering fixed points of the retraction that maps G_i to P_i by mapping vertices in G_i at distance l from a_i to the point on P_i at distance $\min(l, d(a_i, b_i))$ from a_i.

Let H_i for $1 \leq i \leq r$ be a graph consisting of an edge joining two vertices 0 and 1. Considering a retraction on $H_i \Box P_i$ that leaves $0a_i$ and all $1x$ fixed, by mapping all $0x$ with $x \neq a_i$ to $1y$, where y is the neighbor of x on P_i towards a_i. If $y_i x_i$ is the vertex of $H_i \Box G_i$ so constrained, then there is no solution with $y_i = 0$ for all $1 \leq i \leq r$, since such a solution would have $x_i = a_i$ for all $1 \leq i \leq r$. However, for each $1 \leq i \leq r$, there exists a solution with $y_i = 1$ and $y_j = 0$ for all $1 \leq j \leq r$ with $j \neq i$.

In particular, if we require $y_j = 0$ for all $3 \leq j \leq r$ with an appropriate retraction on H_j, we have that (y_1, y_2, y_3) cannot be $(0, 0, 0)$ but can be $(0, 0, 1)$, $(0, 1, 0)$, or $(1, 0, 0)$. Furthermore, for $1 \leq i < j \leq 3$, we can define a retraction on $H_i \Box H_j$ that maps 11 to 00, while leaving $00, 01, 10$ fixed. This enforces $(y_i, y_j) \neq (1, 1)$, so that the only possible values for (y_1, y_2, y_3) are $(0, 0, 1)$, $(0, 1, 0)$, and $(1, 0, 0)$. Therefore, by using fixed points of several nonexpansive mappings, we have defined the one-in-three satisfiability relation on a boolean domain, and the result follows from the NP-completeness of one-in-three satisfiability [7]. $\Box$

Corollary 3.3 *Let $\mathcal{G}$ be a family of graphs. Then the problem of finding a common fixed point of several nonexpansive mappings f_i on a product $G =$*

$G_1\Box G_2\Box\cdots\Box G_n$ of graphs G_i from $\mathcal{G}$ is either polynomial time solvable, or NP-complete even with just two nonexpansive mappings f_1 and f_2 such that f_1 is composed of nonexpansive mappings on $G_{2i-1}\Box G_{2i}$, and f_2 is given by a permutation σ on $1 \leq i \leq n$ and isomorphisms from G_i to $G_{\sigma(i)}$.

Proof. If $\mathcal{G}$ has a majority function, then polynomiality is shown in Theorem 2.2.

If $\mathcal{G}$ does not have a majority function, then in the NP-completeness proof of Theorem 3.2 we only have nonexpansive mappings f^k_{ij} on $G_i\Box G_j$ whose fixed points give S^k_{ij}. We may then make r copies of each G_i, one for each f^k, so that we now have a single nonexpansive mapping f_1 so composed. It remains to force equality between the r copies of G_i given by $G_{i1},\ldots,G_{ir}$, which is achieved by letting $\sigma(il) = i(l+1)$, with $l+1$ obtained modulo r, and f_2 mapping G_{il} to $G_{\sigma(il)}$ giving this equality correspondence for fixed points of f_2. $\Box$

Corollary 3.4 *Let $\mathcal{G}$ be a family of graphs. Then the problem of finding a common fixed point of several retractions f_i on a product $G = G_1\Box G_2\Box\cdots\Box G_n$ of graphs G_i from $\mathcal{G}$ is either polynomial time solvable, or NP-complete even with just retractions on products $G_i\Box G_j$.*

Proof. Modify the definition of a majority function for $\mathcal{G}$ by considering only nonexpansive mappings f on $G_i\Box G_j$ that are retractions. The polynomiality in the presence of a majority function follows as in Theorem 2.2, while the NP-completeness in the absence of a majority follows as in Theorem 3.2 since all nonexpansive mappings explicitly defined in the proof are retractions. $\Box$

4 Chordal graphs

In this section we characterize the families of chordal graphs giving rise to polynomiality or NP-completeness for the problems of fixed points of multiple nonexpansive mappings and of fixed points of multiple retractions. A chordal graph is a graph that does not contain any chordless cycles of length at least four, where a chordless cycle is a cycle where no two nonconsecutive vertices in the cycle are joined by an edge.

Consider a choice of integers $l_1, l_2, l_3 \geq 0$. The graph $R^0 = R^0_{l_1,l_2,l_3}$ has a core S^0 consisting of a single vertex y; in addition, there are three vertices

x_1, x_2, x_3 in R^0, where vertex x_i is joined to the core S^0 by a path p_i of length l_i from x_i to y.

The graph $R^1 = R^1_{l_1,l_2,l_3}$ has a core S^1 consisting of a triangle $y_1y_2y_3$; in addition, there are three vertices x_1, x_2, x_3 in R^1, where vertex x_i is joined to the core S^0 by a path p_i of length l_i from x_i to y_i.

The graph $R^2 = R^2_{l_1,l_2,l_3}$ has a core S^2 consisting of six vertices $y_1, y_2, y_3, z_{12}, z_{23}, z_{31}$ with a triangle $z_{12}z_{23}, z_{31}$ and edges joining y_1 to both z_{12} and z_{31}, joining y_2 to both z_{12} and z_{23}, and joining y_3 to both z_{23} and z_{31}; in addition, there are three vertices x_1, x_2, x_3 in R^2, where vertex x_i is joined to the core S^0 by a path p_i of length l_i from x_i to y_i.

An subgraph H of a graph G is an *isometric subgraph* if for every pair of vertices a, b in H, the distance between a and b is the same in H as in G.

Let G be a chordal graph, and consider a triple of vertices a_1, a_2, a_3 in G. The triple is of *type 0* if G has an isometric subgraph isomorphic to $R^0_{l_1,l_2,l_3}$ with the vertices x_1, x_2, x_3 coinciding with a_1, a_2, a_3 respectively. The triple is of *type 1* if it is not of type 0 and G has an isometric subgraph isomorphic to $R^1_{l_1,l_2,l_3}$ with the vertices x_1, x_2, x_3 coinciding with a_1, a_2, a_3 respectively. The triple is of *type 2* if it is not of type 0 or 1 and G has an isometric subgraph isomorphic to $R^2_{l_1,l_2,l_3}$ with the vertices x_1, x_2, x_3 coinciding with a_1, a_2, a_3 respectively.

Lemma 4.1 *Let G be a chordal graph. Then every triple of vertices a_1, a_2, a_3 in G is of type 0, type 1, or type 2.*

Proof. Let b_1, b_2, b_3 be three vertices in G satisfying $d(a_i, a_j) = d(a_i, b_i) + d(b_i, b_j) + d(b_j, a_j)$ for $i \neq j$. Note that we may always choose $(b_1, b_2, b_3) = (a_1, a_2, a_3)$; choose the vertices b_i so as to minimize $D = d(b_1, b_2) + d(b_2, b_3) + d(b_1, b_3)$. Assume $d(b_1, b_2) \leq d(b_2, b_3) \leq d(b_1, b_3)$. We consider three cases. If $d(b_1, b_2) = 0$, then $d(b_2, b_3) = d(b_1, b_3) = 0$, since in this case $D = 0$ can be achieved by setting b_3 to the common vertex $b_1 = b_2$. The three paths from a_1, a_2, a_3 to $b_1 = b_2 = b_3$ then form an $R^0_{l_1,l_2,l_3}$ isometric subgraph.

If $d(b_1, b_2) = 1$, then $d(b_2, b_3) = d(b_1, b_3) = 1$. Otherwise, if $d(b_1, b_3) = d(b_2, b_3) + 1$, then in this case $D = 0$ can be achieved by setting b_1 and b_3 to the vertex b_2; and if $d(b_2, b_3) = d(b_1, b_3) \geq 2$, then the two paths from b_2 to b_3 and from b_1 to b_3 must be disjoint except for b_3, since otherwise we could move b_3 to the next common vertex and reduce D; furthermore, the ith vertex on the path from b_2 to b_3 can only have an edge to the ith vertex on the path from b_1 to b_3, since otherwise we could use an edge from the ith vertex in one path to the $(i + 1)$th vertex in the other path to modify the second path to go through the ith vertex in the first path, so that the two paths share a vertex other than b_3 as before. The fact that these are the only additional edges for the cycle through b_1, b_2, b_3 implies that the graph has a chordless cycle of even length at least four, contrary to chordality. Thus b_1, b_2, b_3 form a triangle and the three paths from a_1, a_2, a_3 to b_1, b_2, b_3 form an $R^1_{l_1,l_2,l_3}$ isometric subgraph.

Finally, if $d(b_1, b_2) \geq 2$, then $d(b_1, b_2) = d(b_2, b_3) = d(b_1, b_3) = 2$. Otherwise, the only edges joining the path from b_1 to b_2 to the path from b_1 to b_3 must join the ith vertex in one path to the ith vertex in the other path, otherwise we could reduce D by replacing one of the two paths by a path that shares a vertex other than b_1 with the other path; furthermore, only one such value i is possible, otherwise we would have a chordless cycle of even length at least four, contrary to chordality. Thus there is only one edge joining each two of the three paths from b_1 to b_2, from b_1 to b_3, and from b_2 to b_3, and adding three edges to the cycle going through b_1, b_2, b_3 can only add enough chords to forbid chordless cycles if $d(b_1, b_2) = d(b_2, b_3) = d(b_1, b_3) = 2$. In that case, we have paths $b_1c_{12}b_2$, $b_2c_{23}b_3$, and $b_3c_{31}b_1$, with chords forming a triangle $c_{12}c_{23}c_{31}$. These six vertices form the core S^2, and the three paths from a_1, a_2, a_3 to b_1, b_2, b_3 form an $R^2_{l_1,l_2,l_3}$ isometric subgraph, since a shorter distance $d(a_1, c_{23}) = l_1 + 1$ would allow achieving $D = 0$ by setting b_1, b_2, b_3 to the vertex c_{23}, and similarly for $d(a_2, c_{31}) = l_2 + 1$ and $d(a_3, c_{12}) = l_3 + 1$. $\square$

We strengthen this result from isometric subgraphs to retracts.

Lemma 4.2 *Let G be a chordal graph, with a triple of vertices a_1, a_2, a_3 of type j giving an $R^j_{l_1,l_2,l_3}$ isometric subgraph. Then the $R^j_{l_1,l_2,l_3}$ isometric subgraph is a retract of G.*

Proof. Map the vertices at distance $d \leq l_i$ from a_i in G to the corresponding vertex at distance d from a_i on the path from a_i to the core S^j. If $j = 0, 1$, then the remaining vertices of G can be mapped to any vertex in the core S^j. If $j = 2$, then the remaining vertices can be mapped to vertices in the triangle $z_{12}z_{23}z_{31}$ inside the core S^2, since no vertex will be required to map to a vertex adjacent to all three vertices y_1, y_2, y_3 in the core, otherwise such a vertex would give type 0 to the triple a_1, a_2, a_3. $\square$

The following result characterizes the images of products of cores.

Lemma 4.3 *Let f be a nonexpansive mapping on a product $G \Box G'$ of two chordal graphs, with three vertices $x_1x'_1$, $x_2x'_2$, $x_3x'_3$ that are fixed points of f. Suppose the triple x_1, x_2, x_3 is of type j in G and gives an $R^j_{l_1,l_2,l_3}$*

isometric subgraph with core S^j; suppose the triple x'_1, x'_2, x'_3 is of type j' in G' and gives an $R^{j'}_{l_1,l_2,l_3}$ isometric subgraph with core $S^{j'}$. Then the image of $S^j \Box S^{j'}$ under f is a product $U^j \Box U^{j'}$ for two choices of cores U^j and $U^{j'}$ isomorphic to S^j and $S^{j'}$ respectively.

Proof. The vertex y_i (or y if $j = 0$) maximizes $d(x_i, y_i)$ subject to $d(x_i, x_{i'}) = d(x_i, y_i) + d(y_i, x_{i'})$ for $i' \neq i$. Similarly the vertex y'_i (or y' if $j = 0$) maximizes $d(x'_i, y'_i)$ subject to $d(x'_i, x'_{i'}) = d(x'_i, y'_i) + d(y'_i, x'_{i'})$ for $i' \neq i$. It follows that if $f(y_i y'_i) = t_i t'_i$, then $d(t_i, t_{i'}) = d(y_i, y_{i'}) = j$ and $d(t'_i, t'_{i'}) = d(y'_i, y'_{i'}) = j'$.

If $j = 0$ and $j' = 0$ then $S^j \Box S^{j'}$ is a single vertex and the result follows. If $j = 1$ and $j' = 0$ then the three vertices t_1, t_2, t_3 form a triangle and the result follows. If $j = 2$ and $j' = 0$ then the three vertices t_1, t_2, t_3 are at pairwise distance 2, and if we consider the vertices $u_{ij} y' = f(z_{ij} y')$ then the three u_{ij} must be different, otherwise we could set the type j to 0 if two coincide. Thus the six vertices t_i and u_{ij} form a subgraph isomorphic to S^j and the result follows.

If $j = 1$ and $j' = 1$, then the three vertices t_i form a triangle and the three vertices t'_i also form a triangle. It follows that $f(y_i y'_{i'})$ is $t_i t'_{i'}$ or $t_{i'} t'_i$. Furthermore the six-cycle $y_1 y'_2, y_1 y'_3, y_2 y'_3, y_2 y'_1, y_3 y'_1, y_3 y'_2, y_1 y'_2$ implies that either all $f(y_i y'_{i'}) = t_i t'_{i'}$ or all $f(y_i y'_{i'}) = t_{i'} t'_i$, so the result follows.

If $j = 2$ and $j' = 1, 2$, then consider the S^2 given by $y_1 y'_i, y_2 y'_i, y_3 y'_i, z_{12} y'_i, z_{23} y'_i, z_{31} y'_i$; it must map to $t_1 t'_i, t_2 t'_i, t_3 t'_i, u_{12} t'_i, u_{23} t'_i, u_{31} t'_i$ or if $j' = 2$ then possibly also to $t_i t'_1, t_i t'_2, t_i t'_3, t_i u'_{12}, t_i u'_{23}, t_i u'_{31}$. To see this, say if $i = 1$, note that the two paths of length two $y_1 y'_1, z_{12} y'_1, y_2 y'_1$ and $y_1 y'_1, z_{31} y'_1, y_3 y'_1$ must map to paths of length two because they approach $y_2 y'_2$ and $y_3 y'_3$ respectively. Furthermore $z_{12} y'_1$ and $z_{31} y'_1$ must map to different vertices, since $f(y_2 y'_2)$ and $f(y_3 y'_3)$ cannot be approached simultaneously from $f(y_1 y'_1)$. This gives the two possible ways of mapping the triangle $y_1 y'_1, z_{12} y'_1, z_{31} y'_1$, along G or along G'. If $j' = 1$, then in the second case we would have to switch to proceeding along G for the two paths of length two, and the triangles could not be completed. Note then that the triangle $z_{12} y'_1, z_{31} y'_1, z_{23} y'_1$ cannot be mapped to a single edge, else again $f(y_2 y'_2)$ and $f(y_3 y'_3)$ could be approached simultaneously from $f(y_1 y'_1)$. The same argument implies that the triangles $y_2 y'_1, z_{23} y'_1, y_{12} y'_1$ and $y_3 y'_1, z_{23} y'_1, y_{31} y'_1$ must remain as triangles, proving the claim.

It follows again that either we have all $f(y_i y'_{i'}) = t_i t'_{i'}$ or alternatively if $j' = 2$ all $f(y_i y'_{i'}) = t_{i'} t'_i$, by the same six-cycle argument from the case $j = 1$ and $j' = 1$. Thus all $f(y_i y'_{i'}) = t_i t'_{i'}$, $f(y_i z'_{i'i''}) = t_i u'_{i'i''}$, $f(z_{ii'} y'_{i''}) = u_{ii'} t'_{i''}$, $f(z_{ii'} z'_{i''i'''}) = u_{ii'} u'_{i''i'''}$, or alternatively if $j' = 2$ we have all $f(y_i y'_{i'}) = t_{i'} t'_i$,

$f(y_i z'_{i'i''}) = u_{i'i''} t'_i$, $f(z_{ii'} y'_{i''}) = t_{i''} u'_{ii'}$, $f(z_{ii'} z'_{i''i'''}) = u_{i''i'''} u'_{ii'}$.

There is thus only one possible isomorphism from $S^j \Box S^{j'}$ to $U^j \Box U^{j'}$, unless $j = j' = 1$ or $j = j' = 2$, in which case a second isomorphism first exchanges S^j and $S^{j'}$. $\Box$

Let G be a chordal graph and consider a triple a_1, a_2, a_3 of type j giving possibly several different $R^j_{l_1,l_2,l_3}$. If the corresponding possibly different cores S^j share a vertex, then this vertex in the core is said to be a *uniquely determined* vertex.

Theorem 4.4 *Given a family $\mathcal{G}$ of chordal graphs, the problem of finding a common fixed point of several retractions on products of graphs from $\mathcal{G}$ can be solved in polynomial time if all triples of vertices a_1, a_2, a_3 of type j from a graph G in $\mathcal{G}$ give at least one uniquely determined vertex in the corresponding cores S^j, otherwise the problem is NP-complete.*

Proof. If the property in the statement of the theorem holds, then for every triple a_1, a_2, a_3 of type j from a graph G_i in $\mathcal{G}$, define the majority function $g_i(a_1, a_2, a_3) = b$ where b is a uniquely determined vertex in the corresponding core S^j. Given a retraction mapping f on $G_i \Box G_{i'}$ with fixed points $a_1a'_1, a_2a'_2, a_3a'_3$, the image under f of a product of cores $S^j \Box S^{j'}$ gives by Lemma 4.3 such a product of cores which is fixed under f by the definition of a retraction. The fixed $S^j \Box S^{j'}$ must contain $bb' = g_i(a_1, a_2, a_3) g_{i'}(a'_1, a'_2, a'_3)$ since b and b' are uniquely determined vertices and thus must belong to S^j and $S^{j'}$ respectively. Thus bb' is a fixed point of f. Furthermore, if two of a_1, a_2, a_3 are the same vertex a, then this triple is of type 0 with core a and so $g_i(a_1, a_2, a_3) = a$. This proves that the g_i are majority functions, and a polynomial time algorithm follows from Corollary 3.4.

In the other case, suppose three vertices a_1, a_2, a_3 of type j from a graph G in $\mathcal{G}$ do not give a uniquely determined vertex in the corresponding cores S^j. We show that G does not have a majority function. First we observe that G retracts to an $R^j_{l_1,l_2,l_3}$ by Lemma 4.2, so $b = g(a_1, a_2, a_3)$ must belong to this $R^j_{l_1,l_2,l_3}$. Second, the vertex b cannot be a vertex in a core S^j, since every such vertex can be avoided by a different choice of S^j since there is no uniquely determined core vertex. So the only possibility left is that b is on the path from some a_i to a core S^j, say $a_i = a_1$. Consider however a graph P with only two adjacent vertices c and d, so that on P we have $g'(c, d, d) = d$. We can define a retraction f on $R^j_{l_1,l_2,l_3} \Box P$ that leaves every vertex fixed except for the vertices ud, where u is on the path from a_1 to the core S^j (but not in S^j); here we define $f(ud) = vc$, where v follows u on the path from a_1 to the core S_j. It follows that a_1c, a_2d, a_3d are fixed points of f, but bd is not a fixed point of f, so we do not have a majority function. By Corollary 3.4 the problem is NP-complete. $\square$

We derive a similar result for general nonexpansive mappings.

Theorem 4.5 *Given a family $\mathcal{G}$ of chordal graphs, the problem of finding a common fixed point of several nonexpansive mappings on products of graphs from $\mathcal{G}$ can be solved in polynomial time if all triples of vertices a_1, a_2, a_3 of type j from a graph G in $\mathcal{G}$ have all vertices in the corresponding cores S^j uniquely determined, except for the case $j = 2$, where there is either never a y_i that is not uniquely determined, or never a $z_{ii'}$ that is not uniquely determined.*

Proof. Again we define a majority function $g_i(a_1, a_2, a_3) = b$ for triples a_1, a_2, a_3 of type j, by letting b be the uniquely determined vertex y in the core S^0 if $j = 0$, letting b be the uniquely determined vertex y_1 in the core S^1 if $j = 1$, and letting b be the uniquely determined vertex y_1 in the core S^2 if $j = 2$ and there is never a y_i that is not uniquely determined in this case, or letting b be the uniquely determined vertex z_{12} in the core S^2 if $j = 2$ and there is never a $z_{ii'}$ that is not uniquely determined. If $g_i(a_1, a_2, a_3) = b$ and $g_{i'}(a'_1, a'_2, a'_3) = b'$, and $a_1a'_1, a_2a'_2, a_3a'_3$ are fixed points of a nonexpansive f, then bb' is in the product of cores $S^j \square S^{j'}$ and must be left fixed by f by Lemma 4.3. It follows that the g_i are majority functions and a polynomial time algorithm follows from Theorem 2.2.

If the conditions of the theorem are not met, we show that there is no majority function. As in Theorem 4.4, we are left with setting $g_i(a_1, a_2, a_3) = b$ where b is uniquely determined in the corresponding core S^j. However, if say $g_i(a_1, a_2, a_3) = y_1$ in the core S^1, then we must now set say $g_{i'}(a'_1, a'_2, a'_3) = y'_1$ in the core $S^{j'}$ when $j' = 1$, since there is a nonexpansive mapping that exchanges the two cores S^1 and $S^{j'}$ as suggested by Lemma 4.3 so that $f(y_iy'_{i'}) = y_{i'}y'_i$, forcing $i = i'$ to get a fixed point. This thus requires all y_i in S^1 to be uniquely determined. The case with $j = 2$ and the S^2 similary requires all y_i in the S^2 to be uniquely determined, or all $z_{ii'}$ in the S^2 to be uniquely determined, otherwise by exchanging the two cores we would get a vertex that is not a fixed point. The exchanging of the two cores can be performed after retracting $R^j_{l_1,l_2,l_3} \square R^j_{l'_1,l'_2,l'_3}$ to the product of cores $S^j \square S^j$ joined by paths of length $l_1 + l'_1, l_2 + l'_2, l_3 + l'_3$ to $a_1a'_1, a_2a'_2, a_3a'_3$ respectively. This retraction is obtained by first mapping a vertex uv on a path from $y_iy'_{i'}$ to $x_ix'_{i'}$ with $i \neq i'$ to a vertex $u'v'$ on such a path with $d(u, u') = d(v, v')$ and either $u' = y_i$ or $v' = y'_{i'}$. Subsequently, every vertex not on the product of cores $S^j \square S^j$ is on a shortest path from this product to a vertex $x_i x'_i$, and can be mapped according to distance to $x_i x'_i$ along the path of length $l_i + l'_i$ from $x_i x'_i$ and proceeding if necessary into the product of cores. Consequently, the absence of a majority function gives NP-completeness by Theorem 3.2. $\square$

5 Conclusion

We have studied the problem of finding a vertex in a cartesian product of graphs that is a simultaneous fixed point of several given nonexpansive mappings. We found that the problem is polynomial time solvable if the graphs satisfy a farthest point property, and otherwise the simplest counter examples to the farthest point property with two retraction mappings give NP-completeness. The polynomial cases for several given nonexpansive mappings extend to any family of graphs that has a majority function, while families of graphs without a majority function give NP-completeness using just two nonexpansive mappings, thus establishing a dichotomy in the complexity of the problem giving a classification depending on the existence of a majority function for any given class of graphs. A similar classification is obtained for the case of nonexpansive mappings that are retraction mappings. The families of graphs having a majority function for nonexpansive mappings or retractions can be simply characterized in the case of chordal graphs.

We finally note that following the work in [4], one may find in polynomial time a characterization for the set of periodic points of a nonexpansive mapping f given by a black box, which constitutes a retract ot the given product graph. One may similarly obtain as in [4] such a characterization for vertices belonging to fixed subproducts of f. The results in this paper may thus be extended from fixed points and retractions to periodic points and fixed subproducts.

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