Partitioning graphs into ℓ-walks

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1 Introduction and Results

Let $G = (V, E)$ be a connected graph with $|V| = n$ and $|E| = m$, and with d vertices of odd degree; notice that d is necessarily even. An *ℓ-walk* in G is a walk traversing ℓ edges that never traverses the same edge twice but may visit the same vertex several times. Equivalent, and ℓ-walk can be viewed as a connected subset of ℓ edges of G with at most two vertices of odd degree. If $d = 0$, then G is tour-Eulerian, or simly Eulerian, i.e., it has an m-walk that starts and ends in the same vertex. If $d = 2$, then G is walk-Eulerian, i.e., it has an m-walk that starts and ends in the two vertices of odd degree.

In this note, we consider the question of whether the edges of a graph G can be decomposed into edge-disjoint ℓ-walks. We could not determine whether this question is polynomial or NP-complete, except for the easy case $\ell = 2$, and the cases where G is a clique or a tree, or a graph of bounded degree and bounded treewidth. We studied a similar question, namely decomposing the edge of a graph G into copies of a given graph (in particular triangles) in [3].

Theorem 1 *The problem of maximizing the number of edge disjoint 2-walks (induced or not) that can be found in a graph G can be solved in polynomial time.*

Proof. If the 2-walks are not necessarily induced, this is is the problem of finding a maximum matching M in the line graph $L(G)$ of G. If the 2-walks are required to be induced induced, this is is the problem of finding a maximum matching M' in the subgraph $L'(G)$ of the line graph $L(G)$ of G obtained by including edges of $L(G)$ corresponding to adjacent edges of G that are not part of a triangle. □

Theorem 2 *Every connected graph with an even number of edges can be decomposed into edge-disjoint 2-walks.*

Proof. Choose an arbitrary 2-walk. if removing this 2-walk does not disconnect the graph induced by the remaining edges, then proceed by induction. Otherwise if some edge e of the 2-walk disconnects the graph, we can proceed to the two components on either side of e kept with an even number of edges. Finally if the middle vertex v of the 2-walk disconnects the graph, then either the two sides of v have an even number of edges and we proceed by induction, or we can choose a different 2-walk with v in the middle that does not disconnect the graph or that contains an edge that disconnects the graph as before. □

We now infer some necessary conditions for ℓ-walk decomposition.

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Theorem 3 *If the edges of G can be decomposed into edge-disjoint ℓ-walks, then we must have that ℓ divides m. This condition is as well sufficient if G is tour-Eulerian or walk-Eulerian. Otherwise, for $d \geq 4$, an additional necessary condition is that $\ell \leq (2m)/d$.*

Proof. Since the m edges of G must be grouped into sets of ℓ edges, namely the ℓ-walks, it follows that ℓ divides m. If G is tour-Eulerian or walk-Eulerian, then the edges for G form an m walk, and this walk can be decomposed into m/ℓ groups of size ℓ forming ℓ-walds. If $d \geq 4$, we notice that each ℓ-walk in a decomposition has at most two vertices of odd degree, namely the starting end ending vertices of the ℓ-walk, so the d vertices of odd degree must be endpoints of at least $d/2$ many ℓ-walks, giving $m \geq (d/2)\ell$, or equivalently $\ell \leq (2m)/d$. □

The following result solves the problem when G is a clique.

Theorem 4 *The conditions in Theorem 3 are necessary and sufficient if $G = K_n$ is a clique. More precisely, if n is odd, then $d = 0$ and G is tour-Eulerian, so ℓ dividing $m = \binom{n}{2}$ is necessary and sufficient for G to be decomposed into ℓ-walks; if n is even, then $d = n$, and ℓ dividing $m = \binom{n}{2}$ and $\ell \leq (2m)/d = n - 1$ is necessary and sufficient to G to be decomposed into ℓ-walks.*

Proof. If n is odd, then all vertices have even degree $n - 1$, and the result follows immediately from Theorem 3.

If n is even, then all vertices have odd degree $n - 1$, so $d = n$ and the necessary conditions follow from Theorem 3, namely ℓ must divide $m = \binom{n}{2}$ and $\ell \leq 2m/d = n - 1$.

Consider first the limit case $\ell = n-1$. Write $n = 2k$, and label the vertices of $G = K_n$ cyclically as $-k, -k+1, \ldots, -2, -1, 0, 1, 2, \ldots, k-1, k$, where $-k$ and k are the same vertex. Now consider the ℓ-walk $0, 1, -1, 2, -2, \ldots, k-1, -k+1, k$ and consider the k-cyclical rotations of this ℓ-walk starting at $0, 1, 2, \ldots, k-1$. This decomposes $G = K_n$ into $k = n/2$ many ℓ-walks as desired.

Consider next the case where $\ell \leq n - 1$ is odd, which generalizes the case $\ell = n - 1$. Remove from the k $(n-1)$-walks constructed above an even number of edges $n - 1 - \ell$, removing exactly $(n - 1 - \ell)/2$ edges from each end of the walk. After removing these k many ℓ-walks, the remaining graph will be connected (since it will contain all edges alongthe cyclic ordering) and all vertices will have even degree, since the remove ℓ-walks had odd degrees at the endpoints. So the remaining graph is tour-Eulerian, and as long as ℓ divides m it can be decomposed into ℓ-walks as desired.

Finally if $\ell \leq n - 1$ is even, then take $\ell' = \ell - 1$ and find the k many ℓ'-walks as before for ℓ' odd, then add at the beginning of each of these ℓ'-walks a single edge along the cyclic ordering to get an ℓ-walk. Removing these ℓ'-walks gives again a graph with even degrees, and the k edges along the cyclic ordering removed to get ℓ-walks form a path of length k, so removinig these edges creates just two vertices of odd degree. This implies that the graph after removal is walk-Eulerian, since its edges form a connected graph (if $\ell = n - 2$, the connected graph is a path of length k, while if $\ell \leq n - 4$, then the edges that take two steps along the cyclic ordering from two cycles joined by at least one edge that was kept along the cyclic ordering). Therefore as long as ℓ divides m this graph can be decomposed into ℓ-walks as desired. □

The following result gives an algorithm when G is a tree.

Theorem 5 *If G is a tree, then whether the edges G can be decomposed into ℓ-walks is polynomial time solvable even for unbounded ℓ.*

Proof. Root the tree, and proceed from the leafs up. For each edge e encountered, determine whether the subtree underneath can be decomposed into paths of length ℓ plus possibly a remainder $r(e)$ shorter than ℓ. The value of r is the value modulo ℓ of the number of edges underneath e. If underneath e we have edges $e_1, \ldots, e_k$ for k children, the $r(e_i)$ must be paired up so that $r(e_i) + r(e_j) = \ell$, plus one e_i giving $r(e) = r(e_i) + 1$ modulo ℓ, until they all match up at the root. □

The following result follows similarly as well.

Theorem 6 *If G is a graph of bounded degree and bounded treewidth, then whether the edges G can be decomposed into ℓ-walks is polynomial time solvable even for unbounded ℓ.*

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