1. **Problem 1 (20 points)**

   This question is designed to get you used to the notation and mathematical conventions surrounding sets. We strongly suggest working through this problem and double-checking that your answers are correct before starting Problem 2.

   Consider the following sets:

   - $W = \{1, 2, 3, 4\}$
   - $X = \{2, 2, 2, 1, 4, 3\}$
   - $Y = \{1, \{2\}, 1, \{3, 4\}, \{\{3, 4\}\}, \{\{4, 4, 3\}\}\}$
   - $Z = \{1, 3\}$

   Answer the following questions. No explanations are necessary except for the last part.

   a. Which pairs of the above sets, if any, are equal to each other?

   W and X.

   b. Is $Z \in W$? Is $Z \subseteq W$?

   No, $Z \notin W$. Yes. $Z \subseteq W$

   The set $\{1,3\}$ is not contained in W, but the elements 1 and 3 are. Since these are all of the elements in Z, we know that $Z \subseteq W$.

   c. Is $Z \in \mathcal{P}(W)$? Is $Z \subseteq \mathcal{P}(W)$? ($\mathcal{P}(W)$ is the power set of W, that is, the set of all subsets of W. For example, the power set of $\{0, 2, 3\}$ is:

   $\{\emptyset, \{0\}, \{2\}, \{3\}, \{0, 2\}, \{0, 3\}, \{2, 3\}, \{0, 2, 3\}\}$)

   Yes, $Z \in \mathcal{P}(W)$. No, $Z \not\subseteq \mathcal{P}(W)$. $\{1, 3\}$ is a subset of $\{1, 2, 3, 4\}$, so it is in the power set of W. However, neither 1 nor 3 are subsets of $\{1, 2, 3, 4\}$ because they are both elements and not sets.

   d. What is $W \cap Y$? What is $W \cup Y$? $W \cap Y = \{1\}, W \cup Y = \{1, 2, 3, 4, \{2\}, \{3, 4\}, \{\{3, 4\}\}\}$

   e. What is $|X|$? What is $|Y|$? Briefly explain each answer.

   $|X| = 4$ because there are 4 distinct elements in X. $|Y| = 4$ because there are 4 distinct elements in Y.

   Note: Some answers said that $|Y| = 5$. This is not correct, because $\{\{3, 4\}\}$ and $\{\{4, 4, 3\}\}$ are not distinct elements.

2. **Problem 2 (20 points)** The following are logical statements about *finite* sets. Identify the false statements (the ones that are not true for all finite sets A, B, and C), and for each explain why it is false. In each explanation, give concrete examples of sets for which the statement does not hold. No credit will be given for correct identifications without explanations.

   a. $(A \in B \wedge B \in C) \rightarrow A \in C$

   False. Let $A = \{1\}, B = \{\{1\}\}, C = \{\{\{1\}\}\}$. Then $A \in B$ and $B \in C$, but $A \notin C$.

   b. $(A \subseteq B \wedge B \subseteq C) \rightarrow A \subseteq C$

   True. Suppose $A \subseteq B$ and $B \subseteq C$. Let x be some arbitrary element in A. Because $A \subseteq B$, we know that every element in A must also belong to B. Therefore, x must

belong to B. Because $B \subseteq C$, we know that every element of B must also belong to C. Therefore x must belong to C. But x was arbitrary. So every element of A must also belong to C. Therefore, by the definition of subset, $A \subseteq C$.

c. $A \in B \rightarrow \mathcal{P}(A) \in \mathcal{P}(B)$

False. Let $A = \{1\}, B = \{\{1\}\}$. Here we have that $A \in B$.

But $\mathcal{P}(A) = \{\emptyset, \{1\}\}$ and $\mathcal{P}(B) = \{\emptyset, \{\{1\}\}\}$.

Therefore the only elements of $\mathcal{P}(B)$ are the empty set and $\{\{1\}\}$. Notably, the set $\{\emptyset, \{1\}\}$ is not an element of $\mathcal{P}(B)$. Thus, $\mathcal{P}(A) \notin \mathcal{P}(B)$

d. $A \cap \mathcal{P}(A) = A$

False.

Let $A = \{1\}$. Then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$

Elements of A: 1

Elements of $\mathcal{P}(A)$: $\emptyset$, $\{1\}$

We can see that these two sets have no elements in common. Thus, $A \cap \mathcal{P}(A) = \emptyset$. But A does not equal the empty set. So $A \cap \mathcal{P}(A) \neq A$

Note: Some answers missed points on (b.) because they merely asserted that it was true without explanation. We were looking for a proof along the lines of the one presented above, which is the standard way of proving things about subsets.

3. **Problem 3 (10 points)**

   a. Prove that if n is a multiple of 3 then n^2 is a multiple of 3

   Suppose that n is a multiple of 3. Then $n = 3k$ for some integer k. So $n^2 = (3k)^2 = 9k^2 = 3(3k^2)$. Let $p = 3k^2$. We know that p is an integer because k is an integer. Therefore n^2 equals 3 times some integer. So n^2 is a multiple of 3.

   b. Prove that if n is not a multiple of 3 then the remainder when dividing n^2 by 3 is always 1. (Hint: use proof by cases)

   There are two cases to consider:

   Case 1: $n \equiv 1 \bmod 3$.

   Then n can be written as $3k+1$ for some integer k. So $n^2 = (3k+1)^2 = 9k^2+6k+1 = 3(3k^2 + 2k) + 1$. But $(3k^2 + 2k)$ is an integer. So the remainder when dividing n^2 by 3 is 1.

   Case 2: $n \equiv 2 \bmod 3$.

   Then n can be written as $3k + 2$ for some integer k. So $n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. But $(3k^2 + 4k + 1)$ is an integer. So the remainder when dividing n^2 by 3 is 1.

4. **Problem 4 (20 points)**

   Prove that $\sqrt{3}$ is irrational using a method similar to the one we used in class.

   Lemma: Note that we proved in problem **3b** that **if n is not a multiple of 3 then n^2 is not a multiple of 3**. The contrapositive of this statement is that **if n^2 is a multiple of 3, then n is a multiple of 3**.

   We prove by contradiction. Suppose that $\sqrt{3}$ is rational. Then, $\sqrt{3} = p/q$ for some integers p and q that have no common factors (other than 1 and -1).

But $\sqrt{3} = p/q$

$\Rightarrow 3 = (p/q)^2 = p^2/q^2$

$\Rightarrow 3q^2 = p^2$. This implies that p is a multiple of 3 (by our Lemma above). Therefore, $p = 3k$ for some integer k. We substitute p for $3k$ in the previous equation to get:

$3q^2 = (3k)^2 = 9k^2$

$q^2 = 3k^2$. This implies that q is a multiple of 3 (again, by our Lemma above).

But if p and q are both multiples of 3, then they have a common factor, which contradicts our assumption that p and q have no common factors.

5. **Problem 5 (10 points)**

Explain where this proof method goes wrong when it is used to prove that $\sqrt{4}$ is irrational.

We will proceed along the same lines until the proof fails.

We prove by contradiction. Suppose that $\sqrt{4}$ is rational. Then, $\sqrt{4} = p/q$ for some integers p and q that have no common factors (other than 1 and -1).

But $\sqrt{4} = p/q$

$\Rightarrow 4 = (p/q)^2 = p^2/q^2$

$\Rightarrow 4q^2 = p^2$. This implies that p is a multiple of 4.

The proof falls apart here. The fact that p^2 is a multiple of 4 does not imply that p is a multiple of 4. For example, suppose that $p = 2$. Then $p^2 = 4$ is a multiple of 4, but p is not.

Note: Some answers failed to mirror the structure of the proof that $\sqrt{x}$ is irrational by asserting that $4 = p^2/q^2$ implies not that p^2 is divisible by 4, but rather that p^2 is even. Because the proof in part 4 did not rely on even/odd, but rather divisibility by x, this does not show that the proof technique that is used to show that a general number is irrational is faulty. It shows that some other proof technique (which does not work for any case other than $n = 2$) is faulty.

6. **Problem 6 (20 points)**

Suppose that you have a standard 8×8 chessboard with two opposite corners removed:
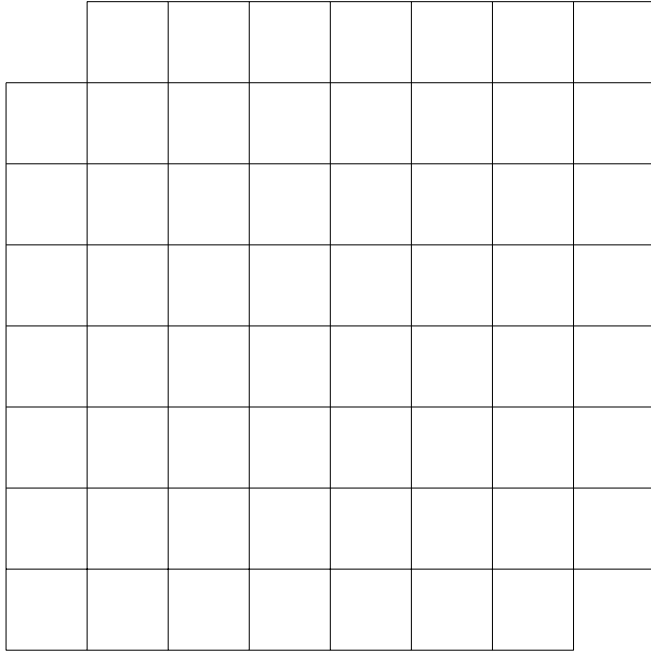
In the course notes (page 62), there's a proof that it's impossible to tile this board using 2×1 dominoes. This question considers what happens if you try to tile the board using right triominoes, L-shaped tiles that look like this:
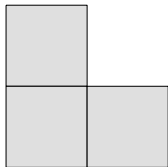
a. Prove that it is impossible to tile an 8×8 board missing two opposite corners with right triominoes.

Each triominoe covers 3 tiles. Therefore, in order for it to be possible to tile the board in triominoes, the number of squares must be a multiple of 3. But 62 is not a multiple of 3. Therefore, it is impossible to tile the board with triominoes.

b. For $n \geq 3$, is it ever possible to tile an $n \times n$ board missing two opposite corners with right triominoes? If so, find a number $n \geq 3$ such that its possible and show how to tile that board with right triominoes. If not, prove that for every $n \geq 3$, it's impossible to tile an $n \times n$ board missing two opposite corners with right triominoes.

There are three cases to consider:

Case 1: $n \equiv 0 \bmod 3$.

Thus $n = 3k$ for some integer k. So the board has $n^2 - 2 = (3k)^2 - 2 = 9k^2 - 2 = 3(3k^2) - 2$ tiles. This cannot be a multiple of 3.

Case 2: $n \equiv 1 \bmod 3$.

Thus $n = 3k + 1$ for some integer k. So the board has $n^2 - 2 = (3k + 1)^2 - 2 = 9k^2 + 6k + 1 - 2 = 3(3k^2 + 2k) - 1$ tiles. This cannot be a multiple of 3.

Case 3: $n \equiv 2 \bmod 3$.

Thus $n = 3k + 2$ for some integer k. So the board has $n^2 - 2 = (3k + 2)^2 - 2 = 9k^2 + 12k + 4 - 2 = 3(3k^2 + 4k) + 2$ tiles. This cannot be a multiple of 3.

Note: many people used two cases, n is a multiple of 3 and n is not a multiple of 3, and used part 3 to show that $n^2 - 2 =$ either $3k - 1$ or $3k - 2$. This also works.