Problem 1

Nim is a family of games played by two players. The game is set up with several of piles of stones. Players take turns removing stones from the piles, such that each move involves removing one or more stones from a single pile. The winner of the game is the player who removes the last stone from play. In other words, if there are no stones available at the start of a player's turn, they have lost the game.

Consider games of Nim which begin with two piles with an equal number of stones. Use induction to show that the player who plays second can always win such a game.

Solution

Let $P(n)$ be "In a game of Nim, if both piles of stones have n stones each and it's the first player's turn, the second player can always win if she plays correctly."

We show that $P(n)$ holds for all $n \in \mathbb{N}$, using strong induction[1] on n.[2]

Base case : If both piles have 0 stones in them, the first player loses according to the rules of the game. Thus, the second player always plays 'correctly' by doing nothing, and wins, so $P(0)$ holds.

Induction hypothesis : Assume that for some nonzero $n \in \mathbb{N}$, $P(i)$ is true for $0 \leq i < n$. This means that in any game of Nim, if both piles of stones have i stones each and it's the rst player's turn, the second player can win if she plays correctly, for $0 \leq i < n$.

Induction step : We show that $P(n)$ holds.

Consider a game of Nim in which there are two piles of stones, A and B, with n stones in each. Without loss of generality, let A be the pile that the first player chooses to remove stones from. The first player must remove k stones from pile A such that $1 \leq k \leq n$, leaving $n - k$ stones in pile A and n stones in pile B. If the second player removes k stones from pile B, this leaves two piles with $n - k$ stones in each. It is now the first player's turn.

By the induction hypothesis, the second player can now win this game because there are two piles with $n - k$ stones in each, $0 \leq n - k < n$, and it is the first player's turn.

Thus, the second player has a strategy by which to win a game of Nim with n stones, and $P(n)$ holds, completing the induction.

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[1]Since it is important to state that strong induction is being used, we deducted points on this problem for neglecting to mention it. We did not deduct points for this on Problem 5.

[2]$0 \in \mathbb{N}$

Problem 2

Consider the following recursive definition of a function $f : \mathbb{N} \to \mathbb{N}$.

$$f(1) = 1$$
$$f(n) = 2f(n-1) + 1$$

Find a non-recursive definition for f, and prove by induction that this definition is correct.

Solution

We can manually derive the following : $f(1) = 1$
$f(2) = 2 \cdot 1 + 1 = 3$
$f(3) = 2 \cdot 3 + 1 = 7$
$f(4) = 2 \cdot 7 + 1 = 15$
$f(2) = 2 \cdot 15 + 1 = 31$

This seems to indicate that $f(n) = 2^n - 1$. We prove this formally by induction.

Define $f' : \mathbb{N} \to \mathbb{N}$ as $f'(n) = 2^n - 1$.[3]

Let $P(n)$ be "$f(n) = f'(n)$". We use induction to show that $f = f'$ by showing that $P(n)$ holds for all $n \in \mathbb{N}$.

Base case : $f(1) = 1$ by definition. $f'(1) = 2^1 - 1 = 1$. $f(1) = f'(1)$, so $P(n)$ holds for $n = 1$.

Induction hypothesis : Assume that $P(n)$ for some $n \in \mathbb{N}$. Equivalently, $f(n) = f'(n)$, or $f(n) = 2^n - 1$.

Induction step : We show that $P(n+1)$ holds, or that $f(n+1) = f'(n+1)$.

$f(n+1) = 2 \cdot f(n) + 1$ by definition of f
$= 2(2^n - 1) + 1$ by the induction hypothesis
$= 2^{n+1} - 2 + 1$ algebra
$= 2^{n+1} - 1$ algebra
$= f'(n+1)$

We have shown that $P(n+1)$ holds, so $f(n) = f'(n) = 2^n - 1$.

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[3] $0 \notin \mathbb{N}$

Problem 3

Prove using induction that $n! < n^n$ for $n > 1$.

Solution

Let $P(n)$ be $n! < n^n$.

Base case : We show that $P(n)$ holds for $n = 2$.
$n! = 2! = 2$
$n^n = 2^2 = 4$
$2 < 4 \rightarrow n! < n^n$ when $n = 2$.

Induction hypothesis : Assume that $P(n)$ holds for some $n \in \mathbb{N}$. That is, $n! < n^n$ for some $n \in \mathbb{N}$.

Induction step : We show that $P(n+1)$ holds. Consider $(n+1)!$.
$(n+1)!$
$= (n+1)n!$ by definition of factorial
$< (n+1)n^n$ by the induction hypothesis
$< (n+1)(n+1)^n$ by Lemma 1
$< (n+1)^{n+1}$

We have shown that $(n+1)! < (n+1)^{n+1}$, thus $P(n+1)$ holds, completing the induction.

Lemma 1[4]
We show that $n^n < (n+1)^n$ for $n > 1$.

Binomial theorem : $(a+b)^k = a^k + ka(k-1)b + \cdots + b^k$

$(n+1)^n$
$= n^n + n \cdot n^{n-1} + ...$ using the Binomial Theorem
$> n^n$

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[4]This is a higher level of detail than we required for full credit.

Problem 4

Show that any wire whose length is exactly x cm ($x \in \mathbb{N}, x \geq 12$) can be cut into some number of pieces, each of which is exactly 4 cm or 5 cm long.

Solution

Let $P(n)$ be "Any wire whose length is exactly n cm can be cut into some number of pieces, each of which is exactly 4 cm or 5 cm long.". Equivalently, $P(n)$ is "There exist natural numbers a and b such that $n = 4a+5b$"

We show that $P(n)$ holds for all n such that $n \in \mathbb{N}$ and $n \geq 12$.

Base cases : We prove $P(12), P(13), P(14), P(15)$ by showing that a wire of length 12 cm, 13 cm, 14 cm or 15 cm can be cut into pieces of exactly 4 cm or 5 cm each.

$12 = 4 \cdot 3 + 5 \cdot 0$
$13 = 4 \cdot 2 + 5 \cdot 1$
$14 = 4 \cdot 1 + 5 \cdot 2$
$15 = 4 \cdot 0 + 5 \cdot 3$

Induction hypothesis : Assume that $P(n)$ holds for some $n \in \mathbb{N}$ such that $n \geq 12$. That is, there exist $a, b \in \mathbb{N}$ such that $n = 4a + 5b$.

Induction step : We show that $P(n + 4)$ holds

$n + 4 = 4a + 5b + 4 = 4(a + 1) + 5b$

So, there exist numbers $a' = (a + 1)$ and $b' = b$ such that $n + 4 = 4a' + 5b'$, and $P(n + 4)$ holds, completing the induction.

Problem 5

For this problem, a *polygon* is a flat, closed shape that has at least 3 vertices. A *diagonal* of a polygon is a straight line joining two non-adjacent vertices of the polygon. A *convex polygon* is a polygon such that any diagonal lies in its interior. Prove by induction that a convex polygon with n vertices has at most $n-3$ non-intersecting diagonals. That is, prove that the cardinality of any set consisting only of non-intersecting diagonals of the polygon is at most $n-3$.

Hint : You may assume without proof that if P is a non-triangular convex polygon with n vertices, then a diagonal of P divides P into an x-vertex convex polygon and a y-vertex convex polygon, such that $n = x+y-2$ and $x < n$ and $y < n$. (the two sub-polygons share exactly two vertices: the vertices of the dividing diagonal). For this problem, your geometric reasoning about diagonals and their intersections (especially how a polygon's diagonals relate to those of its constituent sub-polygons) does not need to be as formal as the rest of your induction proof. While your reasoning needs to be correct, it does not need a lot of detail.

Solution

Let $P(n)$ be "The cardinality of any set consisting only of non-intersecting diagonals of a n-vertex convex polygon is at most $n-3$." We prove the theorem by proving that $P(n)$ holds for all $n \geq 3$. We proceed by strong induction.

Base case : For $n = 3$, the polygon is a triangle. Every vertex in a triangle has zero non-adjacent vertices (since all the vertices are all adjacent to each other). Therefore, there are 0 diagonals, and so the cardinality of any set containing non-intersecting diagonals must be 0. Since $0 \leq n - 3$, $P(3)$ holds.

Induction hypothesis : Assume that $P(i)$ holds for all integers i for which $3 \leq i < n$.

Induction step : We will prove $P(n)$. Consider a polygon P with n (> 3) vertices, and let D be any set consisting of non-intersecting diagonals of P. We need to prove that $|D| \leq n - 3$.

If $D = \emptyset$, since $n > 3$ we can conclude that $|D| = 0 \leq n - 3$, and thus we're done.

If $D \neq \emptyset$, then let d be an arbitrary diagonal in D. Since $n \geq 4$, P is not a triangle, and so d divides P into an x-vertex convex polygon P_x and a y-vertex convex polygon P_y such that $n = x + y - 2$.

Let D_x be the diagonals in $D - \{d\}$ that are diagonals of P_x, and let D_y be the diagonals in $D - \{d\}$ that are diagonals of P_y. Since no two diagonals in D intersect, no two diagonals in D_x or D_y intersect either.

We now informally argue that every diagonal in $D - \{d\}$ must be in exactly one of D_x and D_y. [5]

Assume for the sake of contradiction that there exists a diagonal f that is either in both sets or in neither set.

Case 1 : f is in both sets. If it is in both sets, then the vertices it connects must be in both P_x and P_y. Since the only vertices P_x and P_y share are the ones that d connects, $f = d$, which contradicts the fact that $d \notin D - \{d\}$.

Case 2 : f is in neither set. If it is in neither set, then it must connect a vertex found only in P_x to a vertex found only in P_y. Since P, P_x, and P_y are all convex, f must cross d, which contradicts the fact that f and

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[5]The part of the proof starting from this sentence to "... is in exactly one of D_x and D_y." is more detailed than what you needed. As promised, we were lenient on your geometric reasoning of diagonals. The minimum you needed to assert and informally justify was that the diagonals in $D - \{d\}$ could be partitioned into two sets such that one contains only non-intersecting diagonals of one polygon and the other contains only non-intersecting diagonals of the other polygon. Note that while we argued that every diagonal in $D - \{d\}$ is a diagonal of exactly one sub-polygon, you only needed to argue that every diagonal in $D - \{d\}$ is a diagonal of at least one sub-polygon.

d are non-intersecting diagonals.

Having reached a contradiction in either case, we have shown that every diagonal in $D - \{d\}$ is in exactly one of D_x and D_y.

Since every diagonal in $D - \{d\}$ is in exactly one of D_x and D_y, $|D| = |D_x| + |D_y| + 1$ (the 1 is for d). Since D_x consists of non-intersecting diagonals of P_x and $3 \leq x < n$, by the inductive hypothesis $|D_x| \leq x - 3$. Similarly, $|D_y| \leq y - 3$. Thus, $|D| = |D_x| + |D_y| + 1 \leq (x-3) + (y-3) + 1 = x + y - 5 = (x+y-2) - 3 = n - 3$.

Since our choice of D was arbitrary, we have proven that the cardinality of any set consisting of non-intersecting diagonals of an n-vertex polygon is at most $n - 3$, which completes the induction. $\square$