CS103 HW3

Problem 1

In what follows, if p is a polygon, then let $A(p)$ denote its area.

i Define the relation $=_A$ over the set of all polygons as follows: if x and y are polygons, then $x =_A y$ if and only if $A(x) = A(y)$. Is $=_A$ an equivalence relation? If so, prove it. If not, prove why not.

ii Define the relation $\leq_A$ over the set of all polygons as follows: if x and y are polygons, then $x \leq_A y$ if and only if $A(x) \leq A(y)$. Is $\leq_A$ a partial order? If so, prove it. If not, prove why not.

Solution

(i) Yes, it is an equivalence relation.
Reflexivity: $x =_A x$ for any polygon x since it has the same area as itself.
Symmetry: $x =_A y$ means that polygons x and y have the same area, so $y =_A x$ as well.
Transitivity: $x =_A y$ and $y =_A z$ means that x and y have the same area, and y and z have the same area, so x, y, z all have the same area. Thus $x =_A z$.

(ii) No, $\leq_A$ is not a partial order, as it does not satisfy the Antisymmetry property. Let x and y be any two polygons with the same area (for example, a triangle and a square, both of area 1). Then $x \leq_A y$ since $A(x) \leq A(y)$. Similarly $y \leq_A x$. But x and y are not equal, so Antisymmetry does not hold.

Problem 2

Let $G = (V, E)$ be an undirected graph. The complement of G is the graph $G^c = (V, E')$, that has the same nodes but different edge set E': for any nodes $u, v \in V$, the edge $(u, v) \in E'$ if and only if $(u, v) \notin E$. In other words, the edges in G^c are those not present in G and vice versa.

Prove that for every undirected graph G, at least one of G and G^c is connected. An undirected graph G is called connected if it contains a path between every pair of its vertices.

(Hint: To prove a statement of the form "P or Q," you can instead prove the statement "if P is false, then Q is true." Show that if G isn't connected, then G^c must be connected.)

Solution

Let $G = (V, E)$ be an undirected graph. If G is connected then we are done. Otherwise, G is not connected, so it consists of two or more connected components.

Consider any nodes $u, v \in V$. If u and v belong to different connected components of G, then the edge $(u, v) \notin E$. Therefore, (u, v) must be an edge in G^c.

Otherwise, nodes u and v belong to the same connected component. Consider any node $x \in V$ that belongs to a different connected component than u and v. Then (u, x) and (x, v) are not edges in G, so they must be edges in G^c. Therefore u is connected to v in G^c because we can follow the path u, x, v.

Since our choice of nodes u, v was arbitrary, this establishes that any two nodes in G^c are connected, as required.

Problem 3

A tournament graph is a directed graph with $n \geq 1$ nodes where there is exactly one edge between any pair of distinct nodes and there are no self-loops. Show that if a tournament graph contains a cycle, then it contains a cycle of length 3, that is, a cycle containing 3 edges.

(Hint: consider using a proof by extremal case: consider the smallest cycle in a tournament graph containing a cycle and proceed by contradiction to show that it must have length 3*)*

Solution

Let $G = (V, E)$ be any tournament graph that contains at least one cycle and let C be the smallest cycle in G. C can't have length one, because there are no self-loops in a tournament graph. C also can't have length two, because if $(u, v) \in E$, then $(v, u) \notin E$ because tournament graphs only have one edge between each pair of nodes.

We claim that C has to have length 3 and proceed by contradiction; suppose its length is greater than 3. Let n denote the length of C (by assumption, $n \geq 4$) and let the nodes in C be $v_1, v_2, \ldots, v_n$. Consider the edge between nodes v_2 and v_n. If the edge is of the form (v_2, v_n), then (v_2, v_n, v_1) is a cycle of length 3. This is impossible, because cycle C is the smallest cycle in G and its length is at least four. Therefore, the edge must be (v_n, v_2). Then $(v_2, v_3, ..., v_n, v_2)$ is a cycle in G with length $n - 1$, contradicting that C, the smallest cycle in G, has length n. In either case we reach a contradiction, so our assumption was wrong. Therefore, C must have length 3.

Problem 4

Let G be an undirected graph. The degree of a node v is the number of edges incident to v, i.e. the number of edges with v as one of their endpoints. Prove that G contains two nodes with the same degree.

(Hint: consider two cases: the case where some node has degree 0*, and the case where some node has degree* $n-1$*, where* n *is the number of nodes in* G*)*

Solution

Consider two cases:

Case 1: some node, say v, has degree 0. Then since the remaining nodes are not adjacent to v, their degree is at most $n-2$; thus the possible degrees of any nodes are $0, 1, \ldots, n-2$. Since there are n nodes and $n-1$ possible degrees, by the pigeonhole principle two nodes must have the same degree.

Case 2: some node, say v, has degree $n-1$. Then since the remaining nodes are all adjacent to v, their degree is at least 1; thus the possible degrees of any nodes are $1, 2, \ldots, n-1$. Since there are n nodes and $n-1$ possible degrees, by the pigeonhole principle two nodes must have the same degree.