

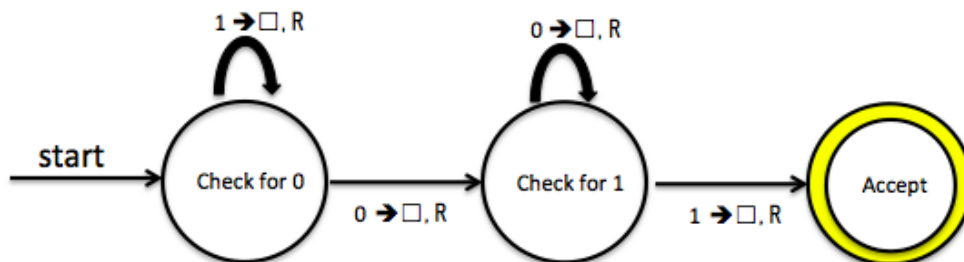
CS103 HW7: Solutions

Problem 1 (20 points)

Let $\Sigma = \{0, 1\}$. Draw the state transition diagram for a Turing machine whose language is $L = \{w \in \Sigma^* \mid w \text{ contains } 01 \text{ as a substring}\}$.

Solution

This Turing machine mimics the DFA for the same language, moving the tape head one step to the right at each step. In the following, assume that missing transitions implicitly cause the TM to reject.



Problem 2 (45 points)

- Prove that there exists a Turing machine M whose language L is decidable, but M is not a decider. This shows that just because a Turing machine's language is decidable, it's not necessarily the case that the Turing machine itself must be a decider.
- Prove that for every language L , there is a decider M that accepts every string in L and another decider M' that rejects every string not in L . Explain why this doesn't prove that every language is decidable.

- (c) Find a pair of languages A and B such that A is a subset of B , B is decidable, but A is not decidable. This shows that a subset of a decidable language is not necessarily decidable, i.e. bigger languages are not necessarily ‘harder.’

Solution

- (a) Let M be the Turing machine that loops indefinitely on all inputs, which has language $L = \emptyset$. L is decidable (the Turing machine that rejects all inputs is a decider for L), but M is not a decider.
- (b) We can set M as the TM that accepts all inputs and M' as the TM that rejects all inputs; then M accepts all strings in L and M' rejects all strings in L , and both are deciders. This doesn't show that L is decidable since the definition of decidability requires the same TM to accept all strings in L and reject all strings not in L , not two different TMs.
- (c) Let A be any undecidable language (e.g. $HALT$), and B be Σ^* . Then B is decidable, and A is a subset of B .

Problem 3 (35 points)

Let $HALT$ be the language $\{\langle M, w \rangle : M \text{ is a TM that halts on } w\}$. Let $ALLHALT$ be the language $\{\langle M \rangle : M \text{ is a TM that halts on all inputs}\}$. Use a reduction from $HALT$ to show that $ALLHALT$ is not decidable.

Solution

We construct a mapping reduction f from $HALT$ to $ALLHALT$, showing that $HALT \leq_M ALLHALT$.

Let $f(\langle M, w \rangle)$ be the TM: “On input x : ignore x and run M on w .” Now M halts on w iff $f(\langle M, w \rangle)$ halts on all inputs, so $\langle M, w \rangle \in HALT$ iff $f(\langle M, w \rangle) \in ALLHALT$, so f is a mapping reduction from $HALT$ to $ALLHALT$, showing that $HALT \leq_M ALLHALT$. Since $HALT$ is not decidable, we conclude that $ALLHALT$ is not decidable.