

Problem 1

Prove using induction that every chess board of size $2^n \times 2^n$, with a single corner tile removed, can be covered using triominoes, the L-shaped tiles defined in HW 1. Consider $n \in \mathbb{N}$, $n > 0$.

Solution:

Base case: Consider $n = 1$. We have a 2×2 board with a corner tile removed, which can be covered using a single triomino. So, the statement holds for $n = 1$.

For the inductive step, assume that the statement holds for $n = k - 1$. We prove that it will also hold for $n = k$.

Proof of the inductive step: For $n = k$, we have a chess board of size $2^k \times 2^k$, with one corner tile removed. We can split this board into 4 parts, three of which are complete chess boards of size $2^{k-1} \times 2^{k-1}$, and one of the same size with a corner tile removed.

The board with the corner tile removed can be covered with triominoes, using the inductive hypothesis.

The other three boards can have all tiles except one corner tile each covered by triominoes, again using the inductive hypothesis. Now we have three boards, of size $2^{k-1} \times 2^{k-1}$, each with one corner tile uncovered. We can arrange the three boards so that the three uncovered tiles join to form a triomino shaped vacancy, and cover it using another triomino.

So, we have proven that if the statement holds for $n = k - 1$, it also holds for $n = k$.

Hence the statement is proven for all n using induction.

Problem 2

Let $G = (V, E)$ be an undirected graph with no self loops. Prove that if the degree of every node in G is at least $|V|/2$, then G is connected.

Solution:

Consider any two nodes, u and v in G . We will prove that there exists a path between u and v in G .

If there exists an edge between u and v , the statement is trivially true.

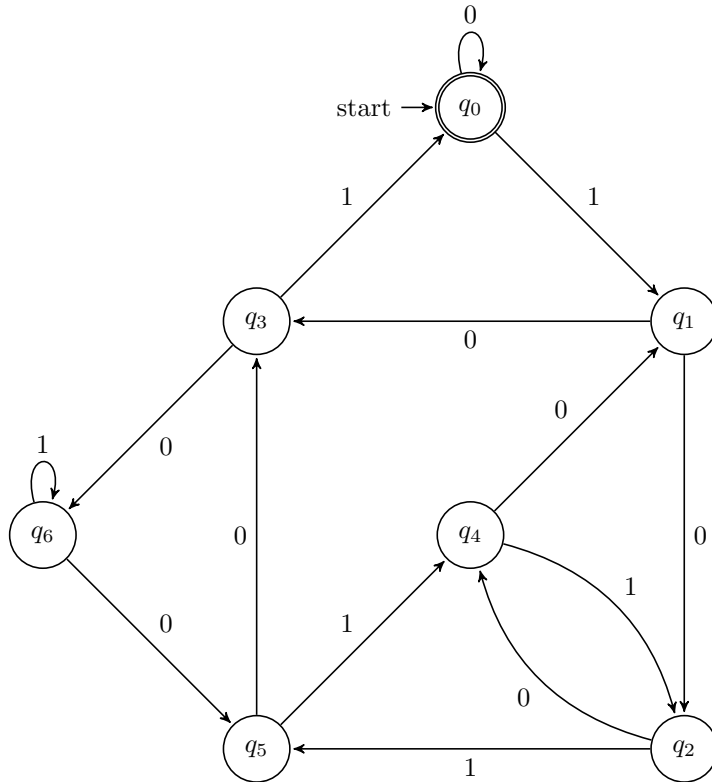
Now, consider the case in which they are not connected. Let $n = |V|$. So, both u and v are connected to at least $(n-1)/2$ other nodes ($n/2$ if n is even). Since there are $n - 2$ nodes other than u and v , and a total of at least $(n - 1)$ outgoing edges from u and v to these nodes, by the pigeonhole principle, there is at least one node w to which both u and v have an edge. (There can be no double edges). So, u and v have a path between them, since the edges are undirected.

Since any two nodes in G have a path between them, G is connected.

Problem 3

Construct a DFA to the language $L = \{s \mid s \text{ represents a binary number divisible by } 7\}$. The alphabet is $\Sigma = \{0, 1\}$.

Solution: Here, we can form a DFA with seven states, each state corresponding to the remainder we get by dividing the currently read string by 7. In the following DFA, q_i corresponds to a remainder of i upon division by 7.



Problem 4

Let $\Sigma = \{0, 1\}$, and define the language $L = \{00^*w00^* \mid w \in \Sigma^*\}$. Prove that L is not regular.

Solution:

Let $x = 01^i0$ and $z = 1^j00$. We can see that $xz \in L$, with $w = 1^j0$. Now, consider $y = 01^j0$. Then, $yz = 01^j01^i00$. We will prove that $yz \notin L$ by contradiction.

Suppose that $yz \in L$. Then, yz can be represented as $0ss0$ for some string s . So, 1^j01^i0 can be represented as ss . s must end with a 0 for this to be true. So, s must be both 1^j0 and 1^i0 , which is not possible. Hence, we cannot have any s for which $yz \in L$. Since we can have infinite equivalence classes, one for each pair of arbitrary numbers (i, j) , by the Myhill Nerode theorem, the language L is not regular.

Problem 5

Let $A_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$.

Using the fact that the language A_{TM} is undecidable, prove that the language

$$L_{101} = \{\langle M \rangle \mid L(M) \text{ contains the string "101"}\}$$

is undecidable.

Solution:

Suppose that L_{101} is decidable. Then, there exists a TM S which decides L_{101} . Using S , we can construct a TM R which decides A_{TM} as follows:

$R =$ " On input $\langle M, w \rangle$:

1. Construct the TM M_w :
 $M_w =$ " On input x :

- (a) Simulate M on w .
 - (b) If M accepts w , accept x .
 - (c) If M rejects w , reject x . ”
2. Run S on $\langle M_w \rangle$.
 3. If S accepts, accept; if R rejects, reject. ”

We can see that if M accepts w , M_w accepts all strings, and the language of M_w contains the string “101”, and hence S accepts $\langle M_w \rangle$. But if M rejects w , S will reject $\langle M_w \rangle$.

We can see that if such an S exists, then A_{TM} is decidable, which is a contradiction. So, no such S exists, and L_{101} is undecidable.

Problem 6

Let $L_l = \{\langle M, w \rangle \mid M \text{ moves its head left at least once when operated on input } w\}$. Can you prove that L_l is undecidable using a proof technique similar to the one used in the previous problem? Prove it if you can, and if not, explain why.

Solution:

Let S be a decider for L_l . Using a proof technique similar to the one used in Problem 5 leads us to form the following TM R which we could claim can decide A_{TM} :

$R =$ “ On input $\langle M, w \rangle$:

1. Construct the TM M'_w :
 $M'_w =$ “ On input x :
 - (a) Simulate M on w .
 - (b) If M accepts w , move left and accept.
 - (c) If M rejects w , move right and accept. ”
2. Run S on $\langle M'_w \rangle$.
3. If S accepts, accept; if R rejects, reject. ”

However, notice that S cannot be used to form a decider for A_{TM} . That is because, M'_w is not guaranteed to not make a left move while simulating M on w . So, even if M rejects w , M'_w can end up making a left move while simulating M . So, S can accept $\langle M'_w \rangle$ even if M rejects w . So, we cannot use S to form a decider for A_{TM} .