CS103 Practice Midterm

Problem 1

For each of the following statements about finite sets, either prove that they are true or give a counterexample to show that they are false.

(i) For any two sets A and B, $B \setminus (B \setminus A) = A$. Here $B \setminus A$ refers to set difference, $B \setminus A = \{x : x \in B \text{ and } x \notin A\}$

(ii) If A, B are sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

(iii) For sets A, B, C, if $A \subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.

(iv) For sets A, B, we have $(A \setminus B) \cup (A \cap B) = A$.

Solution

(i) is false: let $A = \{0\}$ and $B = \{1\}$. Then $B \setminus A = \{1\}$, and $B \setminus (B \setminus A) = \varnothing \neq A$.

(ii) is true: $X \in \mathcal{P}(A)$ means $X \subseteq A$, so $X \subseteq A \cup B$, so $X \in \mathcal{P}(A \cup B)$. Thus $\mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$. Similarly $\mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. Combining the last two statements gives $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

(iii) is false: consider $A = \{1\}, B = \{1, 2\}, C = \{1\}$.

(iv) is true. We will show $(A \setminus B) \cup (A \cap B) \subseteq A$ and $A \subseteq (A \setminus B) \cup (A \cap B)$, which together imply the result.

- Let $x \in (A \setminus B) \cup (A \cap B)$. Then either $x \in A \setminus B$ or $x \in A \cap B$. Either case implies $x \in A$. So $(A \setminus B) \cup (A \cap B) \subseteq A$.
- Now let $x \in A$. Now either $x \in B$ or $x \notin B$. In the former case, we have $x \in A \cap B$. In the latter case, we have $x \in A \setminus B$. Thus in either case $x \in (A \setminus B) \cup (A \cap B)$. So $A \subseteq (A \setminus B) \cup (A \cap B)$.

Problem 2

Prove by induction that if a set S contains n elements, where $n \geq 0$, then its power set $\mathcal{P}(S)$ contains 2^n elements.

Solution

In the base case, for $n = 0$, we have $S = \varnothing$. Then the power set is $\{\varnothing\}$, which indeed contains $2^0 = 1$ elements, so the base case holds.

Assume that the statement is true for $n-1$ elements. Let x be an arbitrary element of S. Now if S contains n elements, then the elements of $\mathcal{P}(S)$ are the subsets of S; separate them into two groups: those that contain x and those that do not.

Those that do not contain x are subsets of the remaining $n-1$ elements of S. By the induction hypothesis, we know that there are 2^{n-1} such subsets.

Those that do contain x are of the form $\{x\} \cup A$, where A is a subset of the remaining $n-1$ elements of S. Again by the induction hypothesis, there are 2^{n-1} such subsets A, and thus 2^{n-1} subsets in this group.

In total, therefore, we have $2^{n-1} + 2^{n-1} = 2^n$ subsets of S, completing the induction.

Problem 3

Show that if an undirected graph G has n vertices, each of degree at least $(n-1)/2$, then the graph is connected.

Solution

By contradiction; assume that G is disconnected. Then G contains at least two connected components, let A and B be different connected components. Since A and B are disjoint, the sum of their sizes is at most n, so one of these components has size at most $n/2$. Let v be any vertex in that component. Then v can only be connected to the remaining nodes in the same component, so its degree is at most $n/2 - 1$, contradicting the fact that the degree of each node is at least $(n-1)/2$.

Problem 4

State which of the following are equivalence relations, and which are partial orders.

(i) Let $xRy = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x|y\}$, where $x|y$ means that x is a factor of y.

(ii) Let S be the set of strings (or sequences of characters, like "cat"). Let $aRb = \{(a, b) \in S \times S : a \text{ and } b \text{ have the same length }\}$.

(iii) Let $xRy = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : |x - y| \leq 1\}$.

Solution

(i) is a partial order.
(ii) is an equivalence relation.
(iii) is not transitive, so it is neither a partial order nor an equivalence relation.

Problem 5

Suppose $\sim$ is a relation on a set A, and that $\sim$ is reflexive and for all $a, b, c \in A$, if $a \sim b$ and $a \sim c$, then $b \sim c$. Show that $\sim$ is an equivalence relation.

Solution

Assume that $a \sim b$. Since also $a \sim a$, the given relationship implies that $b \sim a$. Thus symmetry holds.

Now if $a \sim b$ and $b \sim c$, then also $b \sim a$ and $b \sim c$, so by the given relationship $a \sim c$, so transitivity holds.