Problem Set 8

This problem set is due on **Thursday November 3, by 5:00pm.**

Use the CS172 drop box.

Write **your name and your student ID number** on your solution. Write legibly. The description of your proofs should be as *clear* as possible (which does not mean *long* – in fact, typically, good clear explanations are also short.) Be sure to be familiar with the collaboration policy, and read the overview in the class homepage `www.cs.berkeley.edu/~luca/cs172`.

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1. Define $NR = \{x : K_U(x) < n\}$ as the set of strings that are not Kolmogorov random. Show that NR is recognizable.

2. (Sipser problem 6.20.) Show how to compute the Kolmogorov complexity $K_U(x)$ of a string x with an oracle for A_{TM}.

   The definition of an oracle is given in Sipser definition 6.20 on page 233. An oracle is essentially a subroutine. You could interprete this problem as asking for an algorithm that on input x, computes the descriptive complexity of x, that is, $K_U(x)$, using a subroutine for A_{TM}. On input $\langle M, w\rangle$ the subroutine will return 1 if M accepts w, and 0 otherwise. Whenever you invoke the subroutine on some input $\langle M, w\rangle$, use the terminology "query the A_{TM} oracle on input $\langle M, w\rangle$".

   For instance, the machine S in the proof that $HALT_{\mathsf{TM}}$ is undecidable in Sipser theorem 5.1 (page 188-189) is an example of a algorithm for $HALT_{\mathsf{TM}}$ using an oracle for A_{TM}. In that example, S only uses the A_{TM} subroutine once. In general (and for this problem), you are allowed to invoke the subroutine any number of times, and the oracle queries may be adaptive (that is, the next query may depend on the answers to the previous ones).

3. (Sipser problem 6.13.) Consider the theory $\text{Th}(\mathbb{Z}_5, +, \times)$ defined like the theory $\text{Th}(\mathbb{N}, +, \times)$ except that addition and multiplication are perfomed modulo 5.

   We allow variables $x_1, \ldots, x_n, \ldots$, and

   - for every three variables x_i, x_j, x_k, we have that $x_i + x_j = x_k \pmod 5$ is an expression with free variables x_i, x_j, x_k and that $x_i \times x_j = x_k \pmod 5$ is also an expression with free variables x_i, x_j, x_k;
   - If E_1, E_2 are expressions, having free variables X_1 and X_2 respectively, then $E_1 \vee E_2$ and $E_1 \wedge E_2$ are expressions, having free variables $X_1 \cup X_2$. We also have that $\neg E_1$ is an expression, with free variables X_1.
   - If E is an expression with free variables X, and $x_i \in X$, then $\exists x_i.E$ and $\forall x_i.E$ are expressions with free variables $X - \{x_i\}$.
   - An expression with no free variables is a *statement*.

   For example, the statement $\forall x.\exists y.(y + y = x \pmod 5)$ is true (try it), but the statement $\forall x.\exists y.(y \times y = x \pmod 5)$ is false (consider $x = 2$).

   Show that $\text{Th}(\mathbb{Z}_5, +, \times)$ is decidable.