CS 172 Spring 2007 — Discussion Handout 7

1. **Let's model it!**

   (a) Give a model in which the following sentence is true (such a model is often called the model of the sentence):

   $$\phi_{eq} = \forall x \left[R_1(x,x) \right] \;\wedge\; \forall x,y \left[R_1(x,y) \Leftrightarrow R_1(y,x) \right] \;\wedge\; \forall x,y,z \left[(R_1(x,y) \wedge R_1(y,z) \implies R_1(x,z)) \right]$$

   (b) How does the model change if we add the sentence:

   i. $\forall x \exists y \left[(x \neq y) \;\wedge\; R_1(x,y) \;\wedge\; \forall z (R_1(x,z) \implies x = z \;\vee\; y = z) \right]$

   ii. $\forall x,y \left[\exists u (u \neq x \wedge R_1(u,x)) \;\wedge\; \exists v (v \neq y \wedge R_1(v,y)) \implies R_1(x,y) \right]$

   iii.　$\forall x,y \left[R_1(x,y) \implies \neg R_2(x,y) \right]$
   $\wedge\; \forall x,y \left[\neg R_1(x,y) \implies (R_2(x,y) \oplus R_2(y,x)) \right]$
   $\wedge\; \forall x,y,z \left[(R_2(x,y) \wedge R_2(y,z)) \implies R_2(x,z) \right]$
   $\wedge\; \forall x \exists y \left[R_2(x,y) \right]$

2. **Bigger is better (?)**
   In this problem we try to see if there are sentences that necessarily require large models.

   (a) For a given $k > 0$, give a sentence that requires a model of size at least k.

   (b) Consider the following sentence:

   $$\forall x \exists y R(x,y) \;\wedge\; \forall x,y \neg (R(x,y) \wedge R(y,x)) \;\wedge\; \forall x,y,z (R(x,y) \wedge R(y,z) \implies R(x,z))$$

   i. Show that the natural numbers, with an appropriately chosen R, give a model for this sentence.
   ii. Show that any model for this sentence must necessarily be infinite.