CS 172 Spring 2007 — Discussion Handout 8

1. **How many zeroes?**
   Explain why a Kolmogorov random string of length n (for sufficiently large n) cannot have $n/4$ zeroes and $3n/4$ ones.

2. **Kolmogorov Graphs**
   A graph on n vertices can be represented by string of $n(n-1)/2$ bits (why?). We call a graph G Kolmogorov-random, if the corresponding string is Kolmogorov-random. Show that for sufficiently large n, a Kolmogorov-random graph on n vertices must be connected.

3. **Computable sets cannot be random**
   Let A be any enumerable subset of natural numbers. An infinite binary string $a_0a_1a_2\ldots$ is called the characteristic sequence of A if $a_i = 1 \Leftrightarrow i \in A$. Show that for sufficiently large n

   $$K(a_0a_1\ldots a_n) = O(\log n)$$

4. **The (weak) Prime Number Theorem**
   The prime number theorem says that for large n, the number of prime numbers less than n (denoted by $\pi(n)$) is approximately $n/\ln n$. We can use Kolmogorov complexity to show a weak version of this statement which shows that for infinitely many n, $\pi(n) \geq \frac{n}{\log^2 n}$.

   Let p_i denote the ith prime, so we have $p_1 = 2, p_2 = 3$, etc. Fix any positive integer m (written as a binary string), and let p_k be the largest prime that divides m. Then, we can "describe" m by specifying p_k and m/p_k.

   (a) Show that $K(m) \leq 2\log|k| + |k| + |m/p_k| + O(1)$. (Here, $|k|$ denotes the length of the binary representation of k, and $O(1)$ is a universal constant independent of k and m. We also know that for all binary strings x, y, $K(xy) \leq 2K(x) + K(y) + O(1)$.)

   (b) By picking m to be a Kolmogorov-random string, show that $p_k \leq O(k(\log k)^2)$.

   (c) Show that this gives an n such that $\pi(n) \geq \frac{n}{\log^2 n}$.

   (d) Can you improve this bound to $\frac{n}{\log n(\log\log n)^2}$?