Problem Set 7

This problem set is due on **Wednesday April 4, by 5:00pm.**

Use the CS172 drop box.

Write **your name and your student ID number** on your solution. Write legibly. The description of your proofs should be as *clear* as possible (which does not mean *long* – in fact, typically, good clear explanations are also short.) Be sure to be familiar with the collaboration policy, and read the overview in the class homepage `www.cs.berkeley.edu/~luca/cs172`.

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1. Let A and B be two languages. Then show that:
   (a) If A and B are in NP, then so are $A \cup B$ and $A \cap B$.
   (b) If A and B are NP-complete, then $A \cup B$ and $A \cap B$ *need not be* NP-complete.

2. Let $U = \{\langle M, x, \#^t \rangle | NDTM\ M$ accepts input x within t steps on at least one branch$\}$. Show that U is NP-complete.

3. For a function $g : \mathbb{N} \to \mathbb{N}$, we say a language L is in $\mathbf{SIZE}(g(n))$ if there exists a family of circuits $C_1, C_2, \ldots$ (with C_i having i inputs and one output) such that:
   - $\forall n \in \mathbb{N}$ the size of C_n is at most $g(n)$
   - $\forall x \in \{0,1\}^n$ $x \in L \Leftrightarrow C_n(x) = 1$.

   In the class we saw a proof that $\mathbf{SIZE}(2^{o(n)}) \subsetneq \mathbf{SIZE}(2^n)$ i.e. for every large enough n there exists a function $f : \{0,1\}^n \to \{0,1\}$ that is not computable by circuits of size $2^{o(n)}$. This problem asks you to show such a “separation result” for a smaller function. Show that $\mathbf{SIZE}(n^3/100 \log n) \subsetneq \mathbf{SIZE}(n^3)$.

   Note: You may try (just for fun, this is not part of the homework) generalizing your proof to show that for all $g : \mathbb{N} \to \mathbb{N}$ such that $g(n) = o(2^n/\log n)$ and some constant C, $\mathbf{SIZE}(g(n)/C \log g(n)) \subsetneq \mathbf{SIZE}(g(n))$.

   This result can actually be extended much further. One can even show that $\mathbf{SIZE}(g(n)-1) \subsetneq \mathbf{SIZE}(g(n))$ - every gate counts! But this might take a little longer than the spring break to prove - so you can skip that.