Problem Set 9

This problem set is due on **Wednesday April 25, by 5:00pm.**

Use the CS172 drop box.

Write **your name and your student ID number** on your solution. Write legibly. The description of your proofs should be as *clear* as possible (which does not mean *long* – in fact, typically, good clear explanations are also short.) Be sure to be familiar with the collaboration policy, and read the overview in the class homepage `www.cs.berkeley.edu/~luca/cs172`.

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1. (Sipser 8.25) An undirected graph is bipartite if its nodes can be divided into two sets such that all edges go from a node in one set to a node in the other set. Show that a graph is bipartite iff it does not contain a cycle that contains an odd number of nodes. Let $BIPARTITE = \{\langle G\rangle \mid G \text{ is a bipartite graph}\}$. Show that $BIPARTITE \in \mathbf{NL}$.

2. (Sipser 8.23) Define $UCYCLE = \{\langle G\rangle \mid G$ is an undirected graph that contains a simple cycle$\}$. Show that $UCYCLE \in \mathbf{L}$. (Note: G may not be connected.)

   Hint: We can try to search the tree by always traversing the edges incident on a vertex in lexicographic order i.e. if we come in through the ith edge, we go out through the $(i+1)$th edge or the first edge if the degree is i. How does this algorithm behave on a tree? How about a graph with a cycle?

3. We define the product of two $n \times n$ boolean matrices A and B as another $n \times n$ boolean matrix C such that $C_{ij} = \vee_{k=1}^{n} A_{ik} \wedge Bkj$. (We think of 0 as `false` and 1 as `true` for this problem.)

   (a) Show that boolean matrix multiplication can be done in logarithmic space.

   (b) Using repeated squaring, argue that A^p can be computed in space $O(\log n \log p)$.

   (c) Show that if A is the adjacency matrix of a graph, then $(A^k)_{ij} = 1$ if and only if there is a path of length at most k from the vertex i to vertex j and is 0 otherwise.

   (d) Use the above to give an alternative proof that $\mathbf{NL} \subseteq \mathsf{SPACE}(\log^2 n)$.