Solutions to Problem Set 5

1. Let $B = \{(n, m) \mid$ Every n- state machine M either halts in less than m steps on an empty input, or doesn't halt on an empty input$\}$.

   (a) Show that B is not decidable.

   (b) Show that B is not recognizable.

   [20 + 10 = 30 points] SOLUTION:

   (a) We show that if B is decidable, then we can construct a routine for deciding $HALT_{TM}$ which will be a contradiction. Given an input $\langle M, w\rangle$, we want to decide if M halts on w or not. We first construct a machine N, which just ignores its input and simulates M on w. Hence, N will halt on the empty input if and only if M halts on w.

   Let n be the number of states in N. We can now test if N halts on the empty input as follows:

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k = 1
while (true) {
    if (n,k) ∈ B
        break
    else
        k = k + 1
    }
run N on the empty input for k steps
accept if N halts in at most k steps else reject
```

   Since the number of n-state machines is finite (assuming a fixed alphabet), there must be some maximum k such that all such machines either halt in k steps or run forever. The above algorithm first finds this k and then simply checks if N halts in k steps.

   (b) We show that $\overline{B}$ is recognizable. Since B is not decidable, this implies that B cannot be recognizable.

   $$\overline{B} = \{(n, m) \mid \text{some } n\text{-state machine halts on the empty input after more than } m \text{ steps}\}$$

   Since there are only a finite number of machines with n states, we can simulate all of them in parallel on the empty input. If $(n, m) \in \overline{B}$, then at least one of the machines will halt after more than m steps and we will stop and accept.

2. (Sipser 5.9) Let $T = \{\langle M\rangle \mid M$ is a TM that accepts w^R whenever it accepts $w\}$. Show that T is undecidable.
   [10 points]

   SOLUTION: Let $\mathcal{C} = \{$ languages $L \mid w \in L \Leftrightarrow w^R \in L\}$. Then $L_{\mathcal{C}} = T$. The language 0^* is in $T = L_{\mathcal{C}}$ since $(0^k)^R = 0^k$. 0^* is regular, so there must be some machine for it. So T is not empty. Also $\{01\}$ is finite, so there is a machine for it. And $\{01\}$ is not in T. So T is not everything. By Rice's theorem, T must be undecidable, since it is not everything or empty.

3. (Sipser problem 6.13.) Consider the theory $\mathrm{Th}(\mathbb{Z}_5, +, \times)$ defined like the theory $\mathrm{Th}(\mathbb{N}, +, \times)$ except that addition and multiplication are perfomed modulo 5.

   We allow variables $x_1, \ldots, x_n, \ldots$, and

   - for every three variables x_i, x_j, x_k, we have that $x_i + x_j = x_k \pmod 5$ is an expression with free variables x_i, x_j, x_k and that $x_i \times x_j = x_k \pmod 5$ is also an expression with free variables x_i, x_j, x_k;
   - If E_1, E_2 are expressions, having free variables X_1 and X_2 respectively, then $E_1 \vee E_2$ and $E_1 \wedge E_2$ are expressions, having free variables $X_1 \cup X_2$. We also have that $\neg E_1$ is an expression, with free variables X_1.
   - If E is an expression with free variables X, and $x_i \in X$, then $\exists x_i.E$ and $\forall x_i.E$ are expressions with free variables $X - \{x_i\}$.
   - An expression with no free variables is a *statement*.

   For example, the statement $\forall x.\exists y.(y + y = x \pmod 5))$ is true (try it), but the statement $\forall x.\exists y.(y \times y = x \pmod 5))$ is false (consider $x = 2$).

   Show that $\mathrm{Th}(\mathbb{Z}_5, +, \times)$ is decidable.
   [20 points]

   Solution: Given a formula ϕ, first we write ϕ as $Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \psi(x_1, \ldots, x_n)$ where the Q_i's are quantifiers and ψ has no quantifiers. Now for k from n down to 0, we will define something called I_k with k many inputs. We will compute the value of I_k for each possible input from $\mathbb{Z}_5^k$. Put

   $$I_n(x_1, x_2, \ldots, x_n) = \psi(x_1, x_2, \ldots, x_n)$$

   And for $k > 0$, if $Q_k = \exists$, put

   $$I_{k-1}\left(x_1, x_2, \ldots, x_{k-1}\right) = \bigvee_{i=0}^{4} I_k\left(x_1, x_2, \ldots, x_{k-1}, i\right)$$

   And for $Q_k = \forall$, put

   $$I_{k-1}\left(x_1, x_2, \ldots, x_{k-1}\right) = \bigwedge_{i=0}^{4} I_k\left(x_1, x_2, \ldots, x_{k-1}, i\right)$$

   So I_0 will have no inputs and just be true or false. Output I_0.

   To prove that this works, just see by induction that

   $$\phi \Leftrightarrow Q_1 Q_2 \ldots Q_k I_k$$

   This is automatic for $k = n$ since $\psi = I_n$. And the inductive step works because we are just checking all cases. For $k = 0$ this gives us

   $$\phi \Leftrightarrow I_0$$

   which is what we output.

   So we can decide the theory of $\mathrm{Th}(\mathbb{Z}_5, +, \times)$.