Notes for Lecture 13

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Summary

Today we complete the proof that it is possible to construct a pseudorandom generator from a one-way permutation.

1 Pseudorandom Generators from One-Way Permutations

Last time we proved the Goldreich-Levin theorem.

Theorem 1 (Goldreich and Levin) *Let $f : \{0,1\}^n \to \{0,1\}^n$ be a (t, ϵ)-one way permutation computable in time $r \leq t$. Then the predicate $x, r \mapsto \langle x, r\rangle$ is $(\Omega(t \cdot \epsilon^2 \cdot n^{-O(1)}, 3\epsilon)$ hard core for the permutation $x, r \mapsto f(x), r$.*

A way to look at this result is the following: suppose f is $(2^{\Omega(n)}, 2^{-\Omega(n)})$ one way and computable in $n^{O(1)}$ time. Then $\langle x, r\rangle$ is a $(2^{\Omega(n)}, 2^{-\Omega(n)})$ hard-core predicate for the permutation $x, r \to f(x), r$.

From now on, we shall assume that we have a one-way permutation $f : \{0,1\}^n \to \{0,1\}^n$ and a predicate $P : \{0,1\}^n \to \{0,1\}$ that is (t, ϵ) hard core for f.

This already gives us a pseudorandom generator with one-bit expansion.

Theorem 2 (Yao) *Let $f : \{0,1\}^n \to \{0,1\}^n$ be a permutation, and suppose $P : \{0,1\}^n \to \{0,1\}$ is (t, ϵ)-hard core for f. Then the mapping*

$$x \mapsto P(x), f(x)$$

is $(t - O(1), \epsilon)$-pseudorandom generator mapping n bits into $n + 1$ bits.

Note that f is required to be a permutation rather than just a function. If f is merely a function, it may always begin with 0 and the overall mapping would not be pseudorandom.

For the special case where the predicate P is given by Goldreich-Levin, the mapping would be

$$x \mapsto \langle x, r \rangle, f(x), r$$

PROOF: Suppose the mapping is not $(t-2, \epsilon)$-pseudorandom. There is an algorithm D of complexity $\leq t-2$ such that

$$\left| \Pr_{x \sim \{0,1\}^n} [D(P(x)f(x)) = 1] - \Pr_{\substack{b \sim \{0,1\} \\ x \sim \{0,1\}^n}} [D(bf(x)) = 1] \right| > \epsilon \tag{1}$$

where we have used the fact that since f is permutation, $f(x)$ would be a uniformly random element in $\{0,1\}^n$ when x is such.

We will first remove the absolute sign in (1). The new inequality holds for either D or $1-D$ (i.e. the complement of D), and they both have complexity at most $t-1$.

Now define an algorithm A as follows.

On input $y = f(x)$, pick a random bit $r \sim \{0,1\}$. If $D(r, y) = 1$, then output r, otherwise output $1-r$.

Algorithm A has complexity at most t. We claim that

$$\Pr_{x \sim \{0,1\}^n} [A(f(x)) = P(x)] > \frac{1}{2} + \epsilon$$

so $P(\cdot)$ is not (t, ϵ)-hard core.

To make explicit the dependence of A on r, we will denote by $A_r(f(x))$ the fact that A picks r as its random bit.

To prove the claim, we expand

$$\begin{aligned}
&\Pr_{x,r}[A_r(f(x)) = P(x)] \\
&= \Pr_{x,r}[A_r(f(x)) = P(x) \mid r = P(x)] \Pr[r = P(x)] + \\
&\quad \Pr_{x,r}[A_r(f(x)) = P(x) \mid r \neq P(x)] \Pr[r \neq P(x)]
\end{aligned}$$

Note that $\Pr[r = P(x)] = \Pr[r \neq P(x)] = 1/2$ no matter what $P(x)$ is. The above probability thus becomes

$$\frac{1}{2}\Pr_{x,r}[D(rf(x)) = 1 \mid r = P(x)] + \frac{1}{2}\Pr_{x,r}[D(rf(x)) = 0 \mid r \neq P(x)] \tag{2}$$

The second term is just $\frac{1}{2} - \frac{1}{2}\Pr_{x,r}[D(rf(x)) = 1 \mid r \neq P(x)]$. Now we add to and subtract from (2) the quantity $\frac{1}{2}\Pr_{x,r}[D(rf(x)) = 1 \mid r = P(x)]$, getting

$$\begin{aligned}\frac{1}{2} + \Pr_{x,r}[D(rf(x)) = 1 \mid r = P(x)] - \\ \Big(\frac{1}{2}\Pr[D(rf(x)) = 1 \mid r = P(x)] + \\ \frac{1}{2}\Pr[D(rf(x)) = 1 \mid r \neq P(x)]\Big)\end{aligned}$$

The expression in the bracket is $\Pr[D(rf(x)) = 1]$, and by our assumption on D, the whole expression is more than $\frac{1}{2} + \epsilon$, as claimed.

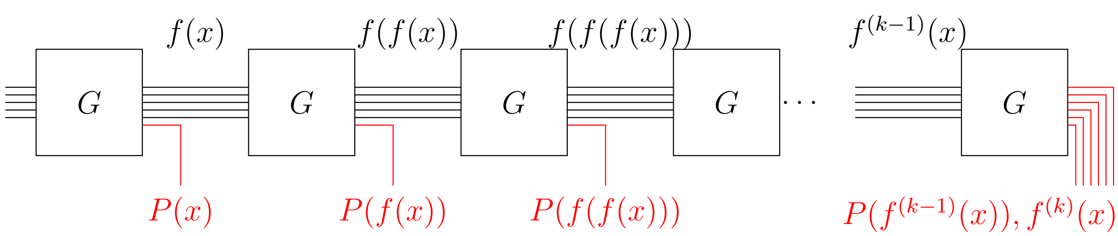
□

The main idea of the proof is to convert something that distinguishes (i.e. D) to something that outputs (i.e. A). D helps us distinguish good answers and bad answers.

We will amplify the expansion of the generator by the following idea: from an n-bit input, we run the generator to obtain $n+1$ pseudorandom bits. We output one of those $n+1$ bits and feed the other n back into the generator, and so on. Specialized to the above construction, and repeated k times the mapping becomes

$$G_k(x) := P(x), P(f(x)), P(f(f(x)), \ldots, P(f^{(k-1)}(x), f^{(k)}(x) \tag{3}$$

This corresponds to the following diagram where all output bits lie at the bottom.



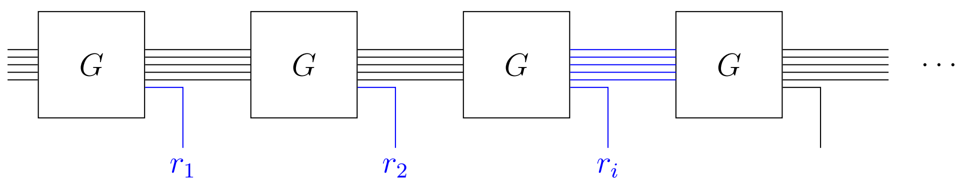
Theorem 3 (Blum-Micali) *Let $f : \{0,1\}^n \to \{0,1\}^n$ be a permutation, and suppose $P : \{0,1\}^n \to \{0,1\}$ is (t,ϵ)-hard core for f and that f, P are computable with complexity r.*

Then $G_k : \{0,1\}^n \to \{0,1\}^{n+k}$ as defined in (3) is $(t - O(rk), \epsilon k)$-pseudorandom.

Proof: Suppose G_k is not $(t-O(rk), \epsilon k)$-pseudorandom. Then there is an algorithm D of complexity at most $t - O(rk)$ such that

$$\left| \Pr_{x \sim \{0,1\}^n}[D(G_k(x)) = 1] - \Pr_{z \sim \{0,1\}^{n+k}}[D(z) = 1] \right| > \epsilon k$$

We will then use the hybrid argument. We will define a sequence of distributions $H_0, \ldots, H_k$, the first is G_k's output, the last is uniformly random bits, and every two adjacent ones differ only in one invocation of G.



More specifically, define H_i to be the distribution where we intercept the output of the first i copies of G's, replace them with random bits, and run the rest of G_k as usual (see the above figure in which blue lines represent intercepted outputs). Then H_0 is just the distribution of the output of G_k, and H_k is the uniform distribution, as desired. Now

$$\begin{aligned}
\epsilon k &< \left| \Pr_{z \sim H_0}[D(z) = 1] - \Pr_{z \sim H_k}[D(z) = 1] \right| \\
&= \left| \sum_{i=0}^{k-1} \left(\Pr_{z \sim H_i}[D(z) = 1] - \Pr_{z \sim H_{i+1}}[D(z) = 1] \right) \right|
\end{aligned}$$

So there is an i such that

$$\left| \Pr_{z \sim H_i}[D(z) = 1] - \Pr_{z \sim H_{i+1}}[D(z) = 1] \right| > \epsilon$$

In both H_i and H_{i+1}, the first i bits $r_1, \ldots, r_i$ are random.

We now define a new algorithm D' that takes as input b, y and has output distribution H_i or H_{i+1} in two special cases: if b, y are drawn from $P(x), f(x)$, then D' has output distribution H_i; if b, y are drawn from (random bit),$f(x)$, then D' has output distribution H_{i+1}. In other words, if b, y are $P(x), f(x)$, D' should output

$$r_1, \ldots, r_i, P(x), P(f(x)), \ldots, P(f^{(k-i-1)}(x)), f^{(k-i)}(x)$$

If b, y are (random bit),$f(x)$, D' should output

$$r_1, \ldots, r_i, r_{i+1}, P(f(x)), \ldots, P(f^{(k-i-1)}(x)), f^{(k-i)}(x)$$

This suggests that D' on input b, y should pick random bits $r_1, \ldots, r_i$ and output $r_1, \ldots, r_i, b, P(y), \ldots, P(f^{(k-i-2)}(y)), f^{(k-i-1)}(y)$.

We have

$$\begin{aligned}
&\left| \Pr_{x \sim \{0,1\}^n} [D'(P(x)f(x)) = 1] - \Pr_{z \sim \{0,1\}^{n+1}} [D'(z) = 1] \right| \\
&= \left| \Pr_{x \sim H_i} [D'(x) = 1] - \Pr_{x \sim H_{i+1}} [D'(x) = 1] \right| \\
&> \epsilon
\end{aligned}$$

and $P(\cdot)$ is not (t, ϵ)-hard core. $\square$

Thinking about the following problem is a good preparation for the proof the main result of the next lecture.

Exercise 1 (Tree Composition of Generators) *Let $G : \{0,1\}^n \to \{0,1\}^{2n}$ be a (t, ϵ) pseudorandom generator computable in time r, let $G_0(x)$ be the first n bits of the output of $G(x)$, and let $G_1(x)$ be the last n bits of the output of $G(x)$.*

Define $G' : \{0,1\}^n \to \{0,1\}^{4n}$ as

$$G'(x) = G(G_0(x)), G(G_1(x))$$

Prove that G' is a $(t - O(r), 3\epsilon)$ pseudorandom generator.