Midterm

To be finished individually. Due on Thursday, February 17, 2011. Submit in class, or by email to trevisan at stanford dot edu

1. Let $G = (V, E)$ be a $d$-regular graph that is 3-colorable and such that there is a 3-coloring in which the color classes have equal size $|V|/3$. Let $A$ be the adjacency matrix and $\frac{1}{d} \cdot M$ be the normalized adjacency matrix. Prove that $M$ has at least two eigenvalues which are smaller than or equal to $-1/2$, that is, $\lambda_{n-1} \leq -1/2$.

[Note: you get partial credit if you prove that there is a negative absolute constant, independent of $|V|$, such that two eigenvalues must be smaller than that constant.]

Give an example in which the bound the tight.

Show that the converse is not true. (That is, give an example of a regular graph that is not 3-colorable but such that at least two eigenvalues of the normalized adjacency matrix are $\leq -1/2$.)

2. Recall that, given two graphs $G = (V, E_G)$ and $H = (V, E_H)$, the non-uniform sparsest cut is

$$
\phi(G, H) = \min_{S \subseteq V} \frac{1}{|E_G|} \cdot \sum_{u,v} A_{u,v} |1_{S}(u) - 1_{S}(v)| \\
\frac{1}{|E_H|} \cdot \sum_{u,v} B_{u,v} |1_{S}(u) - 1_{S}(v)|
$$

where $A$ is the adjacency matrix of $G$ and $B$ is the adjacency matrix of $H$, and the minimum is taken over all sets $S$ that are not empty and are different from $V$.

Consider the following continuous relaxation

$$
\gamma(G, H) = \min_{x \in R^V} \frac{1}{|E_G|} \cdot \sum_{u,v} A_{u,v} |x(u) - x(v)|^2 \\
\frac{1}{|E_H|} \cdot \sum_{u,v} B_{u,v} |x(u) - x(v)|^2
$$

Note that if $H$ is a clique with self-loops and $G$ is regular, then $\gamma(G, H) = 1 - \lambda_2$ and $\phi(G, H) = \phi(G)$. Recall also that $\phi(G) \leq \sqrt{8(1 - \lambda_2)}$, and so we
may hope that, say, when $G$ and $H$ are two arbitrary regular graphs, we have $\phi(G, H) \leq O(\gamma(G, H))$.

Give a counterexample by showing (an infinite family of) regular graphs $G, H$ such that $\phi(G, H) \geq \Omega(1)$ but $\gamma(G, H) = o(1)$.

[Notes: you get full credit even if $G$ and $H$ are not regular. You should be able to get a family of graphs for which $\gamma(G, H) = O(1/n)$ and $\phi(G, H) = \Omega(1)$.

[Hint: Let $G$ be a cycle]