Lecture 13

In which we construct a family of expander graphs.

The problem of *constructing* expander graphs is the task of devising a family $G_n(V_n, E_n)$, such that $\phi(G_n) \geq c$ for a constant $c > 0$ independent of n and such that the graphs G_n are constructible in time polynomial in n and have size $\geq n$. Usually, we are interested in the additional property that the graphs are regular (although we allow parallel edges and self-loops) and have a fixed constant degree independent of n. Ideally, we would like to have a construction with n vertices for every n, although usually the constructions work only for some subset of the integers n. (For applications, it is usually enough if for every n there is an expander whose number of vertices is $\geq n$ and $\leq n^{O(1)}$ in the family.) Such families of graphs have several applications.

There are two main approaches to the construction of expander graphs. In the *algebraic* approach the set of vertices is seen as a group, and the edge set is defined in terms of group operations. These constructions have usually the advantage of being very easy to describe, and to lead to very efficient computations in applications. In particular, it is usually possible, given the $O(\log n)$-bit long description of a vertex v, to compute in time $(\log n)^{O(1)}$ the list of neighbors of v. This efficient "local" computability of the graph enables applications in which one uses an expander of exponential size and, for example, performs short random walks on the graph. The disadvantage of this approach is that the analyses of algebraic constructions tend to be extremely difficult. Today we will see one of the few algebraic constructions that have a relatively easy analysis.

The other approach is *combinatorial* and it relies on graph operations that, given two expanders with n_1 and n_2 vertices, respectively, construct a new expander with $n_1 \cdot n_2$ vertices. One can then start with a small expander, which can be found by brute force, and then construct arbitrarily large expanders by repeated application of the graph operation.

1 The Marguli-Gabber-Galil Expanders

We present a construction of expander graphs due to Margulis, which was the first explicit construction of expanders, and its analysis due to Gabber and Galil. The analysis presented here includes later simplifications, and it follows an exposition of James Lee.

For every n, we construct graphs with n^2 vertices, and we think of the vertex set as $\mathbb{Z}_n \times \mathbb{Z}_n$, the group of pairs from $\{0, \ldots, n-1\} \times \{0, \ldots, n-1\}$ where the group operation is coordinate-wise addition modulo n.

Define the functions $S(a, b) := (a, a+b)$ and $T(a, b) := (a+b, b)$, where all operations are modulo n. Then the graph $G_n(V_n, E_n)$ has vertex set $V_n := \mathbb{Z}_n \times \mathbb{Z}_n$ and the vertex (a, b) is connected to the vertices

$$(a+1, b), (a-1, b), (a, b+1), (a, b-1), S(a, b), S^{-1}(a, b), T(a, b), T^{-1}(a, b)$$

so that G_n is an 8-regular graph. (The graph has parallel edges and self-loops.)

We will prove that there is a constant $c > 0$ such that $\lambda_2(G_n) \geq c$ for every n.

The analysis will be in three steps.

First, we show that $\lambda_2(G_n)$ is bounded from below, up to a constant, by the "spectral gap" of an infinite graph R_n, whose vertex set is $[0, n]^2$. We write "spectral gap" in quote because we will not define a Laplacian and argue about the existence of a spectrum, but just study the infimum of an expression that looks like a Rayleigh quotient, and prove that $\lambda_2(G_n)$ is at least a constant times this infimum. This is proved by showing that for every test function f that sums to zero defined over $\mathbb{Z}_n^2$ we can define a function g over $[0, n]$ whose integral is zero and whose Rayleigh quotient for R_n is the same, up to a constant factor, as the Rayleigh quotient of f for G_n

Then we consider another infinite graph G_∞, with vertex set $\mathbb{Z} \times \mathbb{Z}$, and again define a formal "spectral gap" by considering the infimum of a Rayleigh quotient and we prove that, for every n, the spectral gap of R_n is bounded from below, up to a constant, by the spectral gap of G_∞. This is proved by showing that if f is a test function whose integral is zero and whose Rayleigh quotient for R_n is small, the Fourier transform of f is a test function of small Rayleigh quotient for G_∞.

Finally, we define a notion of expansion for graphs with a countable number of vertices, such as G_∞. We prove that for infinite graphs with a countable set of vertices there is a Cheeger inequality relating expansion and spectral gap, we prove that G_∞ has constant expansion, and we use the Cheeger inequality to conclude that G_∞ has constant spectral gap. (From the previous steps, it follows that R_n, and hence G_n also have spectral gap bounded from below by an absolute constant.)

2 First Step: The Continuous Graph

For every n, we consider a graph R_n with vertex set $[0, n)^2$ and such that every vertex (x, y) is connected to

$$S(x,y), S^{-1}(x,y), T(x,y), T^{-1}(x,y)$$

where, as before, $S(x,y) = (x, x+y)$ and $T(x,y) = T(x+y, y)$ and the operations are done modulo n.

(We are thinking of $[0,n)$ as a group with the operation of addition modulo n, that is, the group quotient $\mathbb{R}/n\mathbb{Z}$, where $n\mathbb{Z}$ is the group of multiples of n with the operation of addition, just like $\mathbb{Z}_n$ is the quotient $\mathbb{Z}/n\mathbb{Z}$.)

Let $\ell_2([0,n)^2)$ be set of functions $f : [0,n)^2 \to \mathbb{R}$ such that $\int_{[0,n)^2} (f(x,y))^2 \,\mathrm{d}x\,\mathrm{d}y$ is well defined and finite. Then we define the following quantity, that we think of as the spectral gap of R_n:

$$\lambda_2(R_n) := \inf_{f \in \ell_2([0,n)^2)\,:\,\int_{[0,n)^2} f = 0} \frac{\int_{[0,n)^2} |f(x,y) - f(S(x,y))|^2 + |f(x,y) - f(T(x,y))|^2 \,\mathrm{d}x\,\mathrm{d}y}{\int_{[0,n)^2} (f(x,y))^2 \,\mathrm{d}x\,\mathrm{d}y}$$

We could define a Laplacian operator and show that the above quantity is indeed the second smallest eigenvalue, but it will not be necessary for our proof.

We have the following bound.

Theorem 1 $\lambda_2(G_n) \geq \frac{1}{12} \cdot \lambda_2(R_n)$.

Proof: Let f be the function such that

$$\lambda_2(G) = \frac{\sum_{c \in \mathbb{Z}_n^2} |f(c) - f(S(c))|^2 + |f(c) - f(T(c))|^2 + |f(c) - f(c + (0,1))|^2 + |f(c) - f(c + (1,0))|^2}{8 \sum_{c \in \mathbb{Z}_n^2} f^2(c)}$$

For a point $(x,y) \in [0,n)^2$, define $floor(x,y) := (\lfloor x \rfloor, \lfloor y \rfloor)$. We extend f to a function $\tilde{f} : [0,n)^2 \to \mathbb{R}$ by defining

$$\tilde{f}(z) := f(floor(z))$$

This means that we tile the square $[0,n)^2$ into unit squares whose corners are integer-coordinate, and that $\tilde{f}$ is constant on each unit square, and it equals the value of f at the left-bottom corner of the square.

It is immediate to see that

$$\int_{[0,n)^2} \tilde{f}^2(z) \,\mathrm{d}z = \sum_{c \in \mathbb{Z}_n^2} f^2(c)$$

and so, up to a factor of 8, the denominator of the Rayleigh quotient of f is the same as the denominator of the Rayleigh quotient of $\tilde{f}$.

It remains to bound the numerators.

Observe that for every $z \in [0,1)^2$, we have that $floor(S(z))$ equals either $S(floor(z))$ or $S(floor(z)) + (0,1)$, and that $floor(T(z))$ equals either $T(floor(z)$ or $T(floor(z)$. Also, $floor(z + (0,1)) = floor(z) + (0,1)$, and the same is true for $(1,0)$. The numerator of the Rayleigh quotient of $\tilde{f}$ is

$$\sum_{c=(a,b)\in\mathbb{Z}_n^2} \int_{[a,a+1)\times[b,b+1)} |\tilde{f}(z) - \tilde{f}(S(z))|^2 + |\tilde{f}(z) - \tilde{f}(T(z))|^2 \, \mathrm{d}z$$

$$= \frac{1}{2} \sum_{c\in\mathbb{Z}_n^2} |f(c)-f(S(c))|^2+|f(c)-f(S(c)+(0,1))|^2+|f(c)-f(T(c))|^2+|f(c)-f(T(c)+(1,0))|^2$$

because for a (x,y) randomly chosen in the square $[a,a+1) \times [b,b+1)$, there is probability $1/2$ that $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ and probability $1/2$ that $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1$.

Now we can use the "triangle inequality"

$$|\alpha - \beta|^2 \le 2|\alpha - \gamma|^2 + 2|\gamma - \beta|^2$$

to bound the above quantity

$$\begin{aligned}
&\le \frac{1}{2} \sum_{c\in\mathbb{Z}_n^2} |f(c) - f(S(c))|^2 + \\
&2|f(c) - f(c + (0,1))|^2 + 2|f(c + (0,1)) - f(S(c) + (0,1))|^2 + \\
&|f(c) - f(T(c))|^2 + \\
&2|f(c) - f(c + (1,0))|^2 + 2|f(c + (1,0)) - f(T(c) + (1,0))|^2
\end{aligned}$$

which simplifies to

$$= \frac{1}{2} \sum_{c\in\mathbb{Z}_n^2} 3|f(c)-f(S(c))|^2+3|f(c)-f(T(c))|^2+2|f(c)-f(c+(0,1))|^2+2|f(c)-f(c+(1,0))|^2$$

which is at most $3/2$ times the numerator of the Rayleigh quotient of f. □

3 Second Step: The Countable Graph

We now define the graph Z of vertex set $\mathbb{Z} \times \mathbb{Z} - \{(0,0)\}$, where each vertex (a,b) is connected to

$$(a, a+b), (a, a-b), (a+b, a), (a-b, a)$$

Note

For a graph $G = (V, E)$ with an countably infinite set of vectors, define $\ell_2(V)$ to be the set of functions $f : V \to \mathbb{R}$ such that $\sum_{v \in V} f^2(v)$ is finite, and define the spectral gap of G as

$$\lambda_2(G) := \inf_{f \in \ell_2(V)} \frac{\sum_{(u,v)\in V} |f(u) - f(v)|^2}{\sum_v f^2(v)}$$

So that

$$\lambda_2(Z) := \inf_{f \in \ell_2(\mathbb{Z}\times\mathbb{Z}-\{(0,0)\})} \frac{\sum_{a,b} |f(a,b) - f(a,a+b)|^2 + |f(a,b) - f(a+b,a)|^2}{\sum_{a,b} f^2(a,b)}$$

We want to show the following result.

Theorem 2 *For every n, $\lambda_2(R_n) \geq \lambda_2(Z)$.*

Proof: This will be the most interesting part of the argument. Let $f \in \ell_2([0,n)^2)$ be any function such that $\int f = 0$, we will show that the Fourier transform $\hat{f}$ of f has a Rayleigh quotient for Z that is at most the Rayleigh quotient of f for R_n.

First, we briefly recall the definitions of Fourier transforms. If $f : [0,n)^2 \to \mathbb{R}$ is such that

$$\int_{z \in [0,n)^2} f^2(z)\, \mathrm{d}z < \infty$$

then we can write the linear combination

$$f(z) = \sum_{c \in \mathbb{Z}\times\mathbb{Z}} \hat{f}(c) \cdot \chi_c(z)$$

where the basis functions are

$$\chi_{a,b}(x,y) = \frac{1}{n} e^{2\pi i \cdot (ax+by)}$$

and the coefficients are

$$\hat{f}(c) = \langle f, \chi_{a,b} \rangle := \int_{[0,n)^2} f(z)\chi_c(z)\, \mathrm{d}z$$

The condition $\int f = 0$ gives

$$\hat{f}(0,0) = 0$$

and the Parseval identity gives

$$\sum_{c \neq (0,0)} \hat{f}^2(c) = \sum_c \hat{f}^2(c) = \int f^2(z)\, \mathrm{d}z$$

and so we have that the denominator of the Rayleigh quotient of f for R_n and of $\hat{f}$ for Z As usual, the numerator is more complicated.

We can break up the numerator of the Rayleigh quotient of f as

$$\int s^2(z)\,\mathrm{d}z + \int t^2(z)\,\mathrm{d}z$$

where $s(z) := f(z) - f(S(z))$ and $t(z) := f(z) - f(T(z))$, and we can use Parseval's identity to rewrite it as

$$\sum_c \hat{s}^2(c) + \hat{t}^2(c)$$

$$= \sum_c |\hat{f}(c) - \widehat{(f \circ S)}(c)|^2 + |\hat{f}(c) - \widehat{(f \circ T)}(c)|^2$$

The Fourier coefficients of the function $(f \circ S)(z) = f(S(z))$ can be computed as

$$\widehat{(f \circ S)}(a,b) = \frac{1}{n} \int f(S(x,y)) e^{2\pi i(ax+by)}$$

$$= \frac{1}{n} \int f(x, x+y) e^{2\pi i(ax+by)}$$

$$= \frac{1}{n} \int f(x, y') e^{2\pi i(ax+by'-bx)}$$

$$= \hat{f}(a-b, b)$$

where we used the change of variable $y' \leftarrow x + y$.

Similarly, $\widehat{(f \circ T)}(a,b) = \hat{f}(a, b-a)$. This means that the numerator of the Rayleigh quotient of f for R_n is equal to the numerator of the Rayleigh quotient of $\hat{f}$ for Z. $\square$

4 Third Step: Proving a Spectral Gap for Z

Now we need to prove that $\lambda_2(Z) \geq \Omega(1)$. We will prove that Z has constant edge expansion, and then we will use a Cheeger inequality for countable graphs to deduce a spectral gap.

Define the edge expansion of a graph $G = (V, E)$ with a countably infinite set of vertices as

$$\phi(G) = \inf_{A \subseteq V, A \text{ finite}} \frac{E(A, \bar{A})}{|A|}$$

Note that the edge expansion can be zero even if the graph is connected.

We will prove the following theorems

Theorem 3 (Cheeger inequality for countable graphs) *For every d-regular graph $G = (V, E)$ with a countably infinite set of vertices we have*

$$\phi(G) \leq \sqrt{2 \cdot d \cdot \lambda_2(G)}$$

Theorem 4 (Expansion of Z) $\phi(Z) \geq 1.25$.

Putting it all together we have that $\lambda_2(Z) \geq \frac{\phi(Z)^2}{2d} > .195$, $\lambda_2(R_n) > .195$, and $\lambda_2(G_n) > .0162$.

4.1 Cheeger inequality for countable graphs

Proof:[Of Theorem 3] This is similar to the proof for finite graphs, with the simplification that we do not need to worry about constructing a set containing at most half of the vertices.

Let $f \in \ell_2(\mathbb{Z}^2)$ be any function. We will show that ϕ is at most $\sqrt{2r}$ where

$$r := \frac{\sum_{(u,v)\in E} |f(u) - f(v)|^2}{\sum_{v\in V} f^2(v)}$$

is the Rayleigh quotient of f.

For every threshold $t \geq t_{\min} := \inf_{v\in V} f^2(v)$, define the set $S_t \subseteq V$ as

$$S_t := \{v : f^2(v) > t\}$$

and note that each set is finite because $\sum_v f^2(v)$ is finite. We have, for $t > t_{\min}$,

$$\phi(G) \leq \frac{E(S_t, \bar{S}_t)}{|S_t|}$$

and, for all $t \geq 0$

$$|S_t| \cdot \phi(G) \leq E(S_t, \bar{S}_t)$$

Now we compute the integral of the numerator and denominator of the above expression, and we will find the numerator and denominator of the Rayleigh quotient r.

$$\int_0^\infty |S_t| \, \mathrm{d}t = \sum_{v\in V} \int_0^\infty I_{f^2(v)>t} \, \mathrm{d}t = \sum_{v\in V} f^2(v)$$

and

$$\int_0^\infty |E(S_t, \bar{S}_t)\, \mathrm{d}t = \sum_{(u,v)\in E} \int_0^\infty I_{t \text{ between } f^2(u), f^2(v)}\, \mathrm{d}t = \sum_{(u,v)} |f^2(u) - f^2(v)|$$

Which means

$$\phi \leq \frac{\sum_{u,v} |f(u) - f(v)|^2}{\sum_v f^2(v)}$$

Now we proceed with Cauchy Swarz:

$$\begin{aligned}
&\sum_{(u,v)\in E} |f^2(u) - f^2(v)| \\
&= \sum_{(u,v)\in E} |f(u) - f(v)| \cdot |f(u) + f(v)| \\
&\leq \sqrt{\sum_{(u,v)\in E} |f(u) - f(v)|^2} \cdot \sqrt{\sum_{(u,v)\in E} |f(u) + f(v)|^2} \\
&\leq \sqrt{\sum_{(u,v)\in E} |f(u) - f(v)|^2} \cdot \sqrt{\sum_{(u,v)\in E} 2f^2(u) + 2f^2(v)} \\
&= \sqrt{\sum_{(u,v)\in E} |f(u) - f(v)|^2} \cdot \sqrt{\sum_{v\in V} 2df(v)^2}
\end{aligned}$$

And we have

$$\phi \leq \frac{\sqrt{\sum_{(u,v)\in E} |f(u) - f(v)|^2} \cdot \sqrt{2d}}{\sqrt{\sum_{v\in V} f(v)^2}} = \sqrt{2d \cdot r}$$

□

4.2 Expansion of Z

After all these reductions, we finally come to the point where we need to prove that something is an expander.

Proof:[Of Theorem 4] Let A be a finite subset of $\mathbb{Z} \times \mathbb{Z} - \{(0,0\}$.

Let A_0 be the set of elements of A that have one 0 coordinate. Let A_1, A_2, A_3, A_4 be the set of elements of A with nonzero coordinate that belong to the 1st, 2nd, 3rd and 4th quadrant. (Starting from the quadrant of points having both coordinates positive, and numbering the remaining ones clockwise.)

Claim 5 $E(A - A_0, \bar{A}) \geq |A - A_0| = |A| - |A_0|$.

PROOF: Consider the sets $S(A_1)$ and $T(A_1)$; both $S()$ and $T()$ are permutations, and so $|S(A_1)| = |T(A_1)| = |A_1|$. Also, $S(A_1)$ and $T(A_1)$ are disjoint, because if we had $(a, a + b) = (a' + b', b')$ then we would have $b = -a'$ while all the coordinates are strictly positive. Finally, $S(A_1)$ and $T(A_1)$ are also contained in the first quadrant, and so at least $|A_1|$ of the edges leaving A_1 lands outside A. We can make a similar argument in each quadrant, considering the sets $S^{-1}(A_2)$ and $T^{-1}(A_2)$ in the second quadrant, the sets $S(A_3)$ and $T(A_3)$ in the third, and $S^{-1}(A_4)$ and $T^{-1}(A_4)$ in the fourth. □

Claim 6 $E(A_0, \bar{A}) \geq 4|A_0| - 3|A - A_0| = 7|A_0| - 3|A|$

PROOF: All the edges that have one endpoint in A_0 have the other endpoint outside of A_0. Some of those edges, however, may land in $A - A_0$. Overall, $A - A_0$ can account for at most $4|A - A_0|$ edges, and we have already computed that at least $|A - A_0|$ of them land into $\bar{A}$, so $A - A_0$ can absorb at most $3|A - A_0|$ of the outgoing edges of A_0. □

Balancing the two equalities (adding 7/8 times the first plus 1/8 times the second) gives us the theorem. □