Midterm

To be finished individually. Due on Wednesday, February 20, 2013. Submit in class, or by email to trevisan at stanford dot edu

1. Let $G = (V, E)$ be a d-regular graph that is 3-colorable and such that there is a 3-coloring in which the color classes have equal size $|V|/3$. Let A be the adjacency matrix and $L := 1 - \frac{1}{d} \cdot A$ be the Laplacian of G. Prove that L has at least two eigenvalues which are greater than or equal to $3/2$, that is, $\lambda_{n-1} \geq 3/2$.

   [Note: you get partial credit if you prove that there is a constant strictly bigger than 1, independent of $|V|$, such that two eigenvalues must be larger than that constant.]

   Give an example in which the bound the tight.

   Show that the converse is not true. (That is, give an example of a regular graph that is not 3-colorable but such that at least two eigenvalues of the normalized adjacency matrix are $\geq 3/2$.)

2. Recall that, given two graphs $G = (V, E_G)$ and $H = (V, E_H)$, consider the normalized non-uniform sparsest cut problem defined as follows:

$$nsc(G, H) := \min_{S \subseteq V} \frac{\frac{1}{|E_G|} \cdot \sum_{u,v} A_{u,v} |1_S(u) - 1_S(v)|}{\frac{1}{|E_H|} \cdot \sum_{u,v} B_{u,v} |1_S(u) - 1_S(v)|}$$

   where A is the adjacency matrix of G and B is the adjacency matrix of H, and the minimum is taken over all sets S that are not empty and are different from V.

   Consider the following continuos relaxation

$$\gamma(G, H) = \min_{x \in R^V} \frac{\frac{1}{|E_G|} \cdot \sum_{u,v} A_{u,v} |x(u) - x(v)|^2}{\frac{1}{|E_H|} \cdot \sum_{u,v} B_{u,v} |x(u) - x(v)|^2}$$

   Note that if H is a clique with self-loops and G is regular, then $\gamma(G, H) = \lambda_2(G)$ and $nsc(G, H) = nsc(G)$ is the normalized uniform sparsest cut problem on G. Recall also that $nsc(G) \leq 2\phi(G) \leq O(\sqrt{8\lambda_2})$, and so we may hope that, say, when G and H are two arbitrary regular graphs, we have $nsc(G,H) \leq O(\sqrt{\gamma(G,H)})$.

Give a counterexample by showing (an infinite family of) regular graphs G, H such that $nsc(G,H) \geq \Omega(1)$ but $\gamma(G,H) = o(1)$.

[Notes: you get full credit even if G and H are not regular. You should be able to get a family of graphs for which $\gamma(G,H) = O(1/n)$ and $nsc(G,H) = \Omega(1)$.]

[Hint: Let G be a cycle]