U.C. Berkeley — Handout N10
CS294: PCP and Hardness of Approximation — February 27, 2006
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Notes for Lecture 10

Review of the last class

The *edge expansion* of graph $G = (V, E)$ is given by

$$h(G) := \min_{\substack{S \subset V \\ |S| \leq |V|/2}} \frac{\text{edges}(S, V - S)}{|S|}$$

G is an expander if $h(G)$ is large.

Suppose G is a d-regular multigraph, and $n = |V|$. Let M be the adjacency matrix of G. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of M. Then $\lambda_1 = d$ and the quantity $d - \lambda_2$ is a measure of the expansion of G. We proved

Theorem 1 $\frac{d-\lambda_2}{2} \leq h \leq \sqrt{2d(d - \lambda_2)}$.

Today we'll see an explicit construction of arbitrarily large graphs of fixed degree, having large edge expansion. (Here and for the rest of today, graph will mean multigraph.)

Let p be a prime and $t < p$. We'll construct a p^2-regular graph $LD_{p,t}$ with p^{t+1} vertices. The vertex set of the graph will be the $t + 1$ dimensional vextor space $\mathbb{F}_p^{t+1}$ over $\mathbb{F}_p$. To specify the edge set, for each $a \in \mathbb{F}_p^{t+1}$ we need to specify p^2 neighbours of a. We will index these neighbours by pairs of elements of $\mathbb{F}_p$.

For each $\alpha, \beta \in \mathbb{F}_p$ the (α, β) th neighbour of a is $a + \beta(1, \alpha, \alpha^2, \ldots, \alpha^t)$. Thus $LD_{p,t}$ is a Cayley graph with generators $\beta(1, \alpha, \alpha^2, \ldots, \alpha^t)$, with $\alpha, \beta \in \mathbb{F}_p$.

Let $M_{p,t}$ be the adjacency matrix of $LD_{p,t}$. We want to compute a bound on the second eigenvalue of $M_{p,t}$. Our approach will be to construct a complex-valued orthogonal basis of eigenvectors, and use these to infer bounds on the eigenvalues.

Note: Since $M_{p,t}$ is real valued and symmetric, its eigenvalues are all real and may be sorted as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, where $n = p^{t+1}$ If $x_1, \ldots, x_n$ are corresponding complex-valued orthogonal eigenvectors for $M_{p,t}$ then $\text{Re}(x_1), \ldots, \text{Re}(x_n)$ are corresponding real eigenvectors. They need not be orthogonal however, unless the eigenvalues are distinct.

Notation: We will index the coordinates of vectors in $\mathbb{C}^n = \mathbb{C}^{\mathbb{F}_p^{t+1}}$ by elements of $\mathbb{F}_p^{t+1}$. Thus, for $x \in \mathbb{C}^n$, $b \in \mathbb{F}_p^{t+1}$, $x(b)$ will denote the b th coordinate of x.

Let ω be a primitive p th root of unity, *i.e.* $\omega = e^{2\pi i/p}$. We want to define $n = p^{t+1}$ eigenvectors, $x_1, \ldots, x_n$. Again, we'll index them by elements of $\mathbb{F}_p^{t+1}$. We'll define the eigenvectors as follows: For each $a \in \mathbb{F}_p^{t+1}$, let $x_a \in \mathbb{C}^{\mathbb{F}_p^{t+1}}$ be defined by $x_a(b) = \omega^{\sum_j a_j b_j}$.

Claim 2 *The vectors x_a, $a \in \mathbb{F}_p^{t+1}$ are orthogonal.*

PROOF: Recall that for $x, y \in \mathbb{C}^n$ the inner product is defined by $(x, y) = \sum_b x(b)\overline{y(b)}$, where $\overline{y(b)}$ is the complex conjugate of $y(b)$.

We will need the fact that $\sum_{\gamma=0}^{p-1} \omega^\gamma = 0$. If $\alpha \in \{1, 2, \ldots, p-1\}$ then the numbers $(\omega^{\alpha\gamma})_{\gamma=0}^{p-1}$ are a permutation of $(\omega^\gamma)_{\gamma=0}^{p-1}$. It follows that in this case also $\sum_{\gamma=0}^{p-1} \omega^{\alpha\gamma} = 0$.

Consider x_a and x_b where $a \neq b$. We have

$$\begin{aligned}
\sum_{c \in \mathbb{F}_p^{t+1}} x_a(c)\overline{x_b(c)} &= \sum_{c \in \mathbb{F}_p^{t+1}} \omega^{(\sum_j a_j c_j)} \omega^{(-\sum_j b_j c_j)} \\
&= \sum_{c \in \mathbb{F}_p^{t+1}} \omega^{\sum_j (a_j - b_j) c_j} \\
&= \sum_{c_0 \in \mathbb{F}_p} \cdots \sum_{c_t \in \mathbb{F}_p} \prod_{j=0}^{t} \omega^{(a_j - b_j) c_j} \\
&= \prod_{j=0}^{t} \sum_{c_j \in \mathbb{F}_p} \omega^{(a_j - b_j) c_j} \\
&= 0
\end{aligned}$$

The last equality follows because $a \neq b$ means that for some j, $a_j - b_j \neq 0$. The corresponding factor becomes a sum of all powers of a primitive p root of unity, and as has already mentioned, this is zero. Thus $x_a \perp x_b$. $\square$

Claim 3 *For each $a \in \mathbb{F}_p^{t+1}$, x_a is an eigenvector of $M_{p,t}$.*

PROOF: Fix $a \in \mathbb{F}_p^{t+1}$. For any $b \in \mathbb{F}_p^{t+1}$ we have

$$\begin{aligned}
(x_a M_{p,t})(b) &= \sum_c x_a(c) M_{p,t}(c, b) \\
&= \sum_{\alpha, \beta \in \mathbb{F}_p} x_a(b + \beta(1, \alpha, \alpha^2, \ldots, \alpha^t)) \\
&= \sum_{\alpha, \beta \in \mathbb{F}_p} x_a(b) x_a(\beta(1, \alpha, \alpha^2, \ldots, \alpha^t)) && \text{since } x_a(b + c) = x_a(b) x_a(c) \\
&= x_a(b) \underbrace{\sum_{\alpha, \beta \in \mathbb{F}_p} x_a(\beta(1, \alpha, \alpha^2, \ldots, \alpha^t))}_{\lambda_a} \\
&= \lambda_a x_a(b)
\end{aligned}$$

Thus x_a is an eigenvector. $\square$

Note that in proving Claim 3 we have also obtained a closed form expression for the eigenvalues of $M_{p,t}$, namely for $a \in \mathbb{F}_p^{t+1}$,

$$\lambda_a = \sum_{\alpha,\beta \in \mathbb{F}_p} \omega^{\sum_{j=0}^{t} a_j \beta \alpha^j} = \sum_{\alpha,\beta \in \mathbb{F}_p} \omega^{\beta P_a(\alpha)}$$

where P_a denotes the polynomial $P_a(X) = \sum_{j=0}^{t} a_j X^j$ over $\mathbb{F}_p$.

If $a = (0,\ldots,0)$ then $x_a = (1,\ldots,1)$ and $\lambda_a = p^2$, which is the degree of the graph.

If $a \neq (0,\ldots,0)$ then P_a is a non-zero polynomial of degree at most t. Hence it has at most t roots. We have

$$\begin{aligned}
|\lambda_a| &= \left| \sum_{\alpha} \sum_{\beta} \omega^{\beta P_a(\alpha)} \right| \\
&\leq \left| \sum_{\alpha : P_a(\alpha)=0} \sum_{\beta} \omega^{\beta P_a(\alpha)} \right| + \left| \sum_{\alpha : P_a(\alpha)\neq 0} \sum_{\beta} \omega^{\beta P_a(\alpha)} \right| \\
&= \left| \sum_{\alpha : P_a(\alpha)=0} \sum_{\beta} \omega^{0} \right| + \left| \sum_{\alpha : P_a(\alpha)\neq 0} \sum_{\beta} (\omega^{P_a(\alpha)})^{\beta} \right| \\
&\leq pt + 0
\end{aligned}$$

where the second sum is zero once again because it is the sum of all powers of ω.

Thus each of the other eigenvalues is at most pt (even in absolute value). In particular, $\lambda_2 \leq pt$. Since $d = p^2$ we have $d - \lambda_2 \geq p(p-t) > 0$. In summary, we have constructed a graph with p^{t+1} vertices, degree p^2 and edge expansion at least $p(p-t)/2$. Of course, since t is required to be smaller than p, we do not get arbitraily large graphs of fixed degree and large edge expansion this way. In order to achieve this goal, we will need another construction.

Zig Zag product Given graphs G and H of compatible sizes, with small degree and large edge expansion, the zig zag product $G \textcircled{z} H$ is a method of constructing a larger graph also with small degree and large edge expansion. We will need

- G a D-regular graph on n vertices, with $\lambda_2(G) \leq \alpha D$
- H a d-regular graph on D vertices, with $\lambda_2(H) \leq \beta d$

We will get

- $G \textcircled{z} H$ a d^2-regular graph on nD vertices, with $\lambda_2(G \textcircled{z} H) \leq (\alpha + \beta + \beta^2) d^2$.

We will see the construction and analysis of the zig zag product in the next class.

For the remainder of today, we'll see how to use the zig zag product to construct arbitrarily large graphs of fixed degree with large edge expansion.

Fix a large enough constant d. ($1369 = 37^2$ will do.) Construct a d-regular graph H on d^4 vertices with $\lambda_2(H) \leq d/5$. (For example $LD_{37,7}$ is a degree 37^2 graph on $37^{(7+1)} = (37^2)^4$ vertices with $\lambda_2 \leq 37 \times 7 < 37^2/5$.)

For any graph G, let G^2 represent the graph on the same vertex set whose edges are the paths of length two in G. Thus G^2 is the graph whose adjacency matrix is the square of the adjacency matrix of G. Note that if G is r-regular then G^2 is r^2-regular

Using the H from above we'll construct inductively, a family of progressively larger graphs, all of which are d^2-regular and have $\lambda_2 \leq d^2/2$.

Let $G_1 = H^2$. For $k \geq 1$ let $G_{k+1} = (G_k^2)ⓩH$.

Theorem 4 *For each $k \geq 1$, G_k has degree d^2 and $\lambda_2(G_k) \leq d^2/2$.*

Proof: We'll prove this by induction.
Base case: $G_1 = H^2$ is d^2-regular. Also, $\lambda_2(H^2) = (\lambda_2(H))^2 \leq d^2/25$.

Inductive step: Assume the statement for k, *i.e.* G_k has degree d^2 and $\lambda_2(G_k) \leq d^2/2$. Then G_k^2 has degree $d^4 = |V(H)|$, so that the product $(G_k^2)ⓩH$ is defined. Moreover, $\lambda_2(G_k^2) \leq d^4/4$. Applying the construction, we get that G_{k+1} has degree d^2 and $\lambda_2(G_{k+1}) \leq (\frac{1}{4} + \frac{1}{5} + \frac{1}{25})d^2 = \frac{49}{100}d^2$
This completes the proof. $\Box$

Finally note that G_k has d^{4k} vertices.