U.C. Berkeley — Handout N12
CS294: PCP and Hardness of Approximation — March 6, 2006
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Notes for Lecture 12

In the last lecture we described the construction of the zigzag product of graphs and completed the analysis of its expansion. This was the final building block for our construction of arbitrarily large d-regular expanders, for a constant d. In particular, we had $d = 37^2$ and achieved $\lambda_2 \leq \lambda < d/2$.

Notice that, if we needed a larger expansion, we could always recur to graph powering. Given G, with adiacency matrix $M(G)$ and second largest eigenvalue $\lambda_2(G)$, we would then have $M(G^k) = [M(G)]^k$ and $\lambda_2(G^k) = [\lambda_2(G)]^k$. In the particular case of our construction, k-powering would produce a graph with eigenvalue $\lambda_2 < (\frac{d}{2})^k$ and degree d^k, yielding an arbitrarily large gap. By the bounds on expansion of Lecture 9, Theorem 1, we obtain a lowerbound on the expansion of $d^k(1 - \frac{1}{2^k})$, which is larger than 1 already for k=1, and it is hence sufficient for the reduction we presented in Lecture 3 from MAX3SAT to MAX3SAT with bounded occurence of variables.

The following claim (Theorem 3, Lecture 5) was crucial in our amplification of the gap for a PCP-verifier, using random walks:

Theorem 1 *There exists d such that for every n there is a d-regular graph $G = (V, E)$ with n vertices such that for every $B \subset V$ s.t. $|B| \leq \frac{|V|}{2} = \frac{n}{2}$,*

$$\mathbf{Pr}[\textit{random walk in } G \textit{ of length } k \textit{ is completely contained in } B] \leq \left(\frac{2}{3}\right)^k$$

Here and in the rest of this lecture, a random walk has starting point uniform over V and at each step chooses with equal probability one of the d edges incident to the current vertex.

Today we are going to establish Theorem 1 by showing the more general result:

Theorem 2 *Let $G = (V, E)$ be a d-regular expander with $\lambda(G) \leq \alpha d$ for $\alpha < 1$. For $B \subset V$, let $\beta = \frac{|B|}{|V|}$. Then:*

$$\mathbf{Pr}[\textit{random walk in } G \textit{ of length } k \textit{ is completely contained in } B] \leq (\alpha + \beta)^k$$

Note that the starting point of the walk needs to be random. The result does not hold for an arbitrary deterministic starting vertex in B, as such vertex could lie deep within B and require many steps of the random walk to exit B.

Theorem 2 is sufficient to ensure Theorem 1 if we have $\alpha \leq \frac{1}{6}$, which we can obtain from our construction of expander graphs through powering, as described above. Moreover, amplifying the spectral gap of G through graph powering, we can make the probability in Theorem 1 as small as $\frac{1}{2} + \epsilon$ for any $\epsilon > 0$.

Before stating the proof for Theorem 2, we introduce the following definitions, which apply to a d-regular graph $G = (V, E)$, $|V| = n$ with adiacency matrix M.

Definition 3 (Transition Matrix) *The transition matrix P is the matrix $P = \frac{1}{d}M$.*

Each entry $P(u, v)$ of the matrix P equals the number of edges between u and v divided by the degree, i.e. the probability that a random walk currently at u proceeds to v in the next step. Similary, $P^2(u, v) = \sum_{w \in V} P(u, w)P(w, v)$ equals the probability that a random walk at u moves to v in two steps. By induction, we can generalize to claim that $P^k(u, v)$ equals the probability of a transition from u to v in k steps. Finally, if M had eigenvalues $d, \lambda_2, \cdots, \lambda_n$, P has eigenvalues $1, \frac{\lambda_2}{d}, \cdots, \frac{\lambda_n}{d}$.

Consider now a vector $x \in R^V$ representing a probability distribution over V, i.e. for all $u \in V$ $x(u) \geq 0$, $\sum_{v \in V} x(v) = 1$. Then $xP = \sum_{v \in V} x(u)P(u, v)$ equals the distribution vector derived from sampling a starting point from distribution x and taking one step of the random walk defined by P. Similarly xP^k yields the distribution vector of a walk starting at a vertex sampled from x and taking k steps according to P.

Definition 4 *$\pi_u = (\frac{1}{n}, \cdots, \frac{1}{n})$ is the vector representing the uniform distribution over vertices.*

Definition 5 *Let B be the matrix:*

$$B(u, v) = \begin{cases} 1 & u = v, u \in B \\ 0 & \textit{otherwise} \end{cases}$$

In words, the multiplication xB sets all entries of x not in B to 0.

Proof

Let $v_0, \cdots, v_k$ be the vertices visited by the random walk under consideration.

Claim 6

$$\mathbf{Pr}[\textit{random walk of length } k \textit{ is completely contained in } B] = ||\pi_u B(PB)^k||_1$$

Proof: Consider $\mu B(PB)^k(v)$ for $v \in B$ and any starting distribution μ. We claim this equals the probability that a random walk of length k, with starting distribution μ, is completely contained in B and $v_k = v$. This is trivially true for $k = 0$ and, by induction:

$$\begin{aligned}
\mu B(PB)^k(v) &= [\mu B(PB)^{k-1}][PB(v)] = \\
&= \sum_{w \in B} \mathbf{Pr}[v_0, \cdots, v_{k-1} = w \in B]\, \mathbf{Pr}[v_k = v \in B | v_{k-1} = w] = \\
&= \mathbf{Pr}[v_0, \cdots, v_k = v \in B]
\end{aligned}$$

Applying this to $\mu = \pi_u$, we obtain:

$$\mathbf{Pr}[\text{r.w. of length } k \text{ is completely contained in } B] = \sum_{v \in V} \pi_u B(PB)^k(v) = ||\pi_u B(PB)^k||_1$$

as required. $\square$

The following claim helps us to bound this quantity.

Claim 7 *For every $x \in R^V$:*

$$||xBPB||_2 \leq (\alpha + \beta)||x||_2$$

The main theorem follows directly from this claim as:

$$||\pi_u B(PB)^k||_1 \leq \sqrt{n}||\pi_u B(PB)^k||_2 \leq \sqrt{n}||\pi_u (BPB)^k||_2$$

where the first inequality follows from Cauchy-Schwarz and the second is due to the fact that $BB = B$ as B is diagonal with entries in $0, 1$. By applying the former claim k times we then obtain:

$$\begin{array}{lcl}
lcl||\pi_u B(PB)^k||_1 \leq & \sqrt{n}||\pi_u (BPB)^k||_2 & \leq \sqrt{n}(\alpha+\beta)||\pi_u (BPB)^{k-1}||_2 \leq \\
= & \cdots & \leq \sqrt{n}(\alpha+\beta)^k ||\pi_u||_2 = \sqrt{n}(\alpha+\beta)^k \dfrac{1}{\sqrt{n}} = (\alpha+\beta)^k
\end{array}$$

We are then left with proving Claim 7.

PROOF: In a fashion similar to that in the proof of the spectral gap of the zigzag product, we decompose xB in two components. The first will be constant in every entry and will have its magnitude reduced by β through multiplication by B. The second, perpendicular to π_u, will be shortened by α through multiplication by P. Formally, let $xB = (xB)_{||} + (xB)_{\perp}$, where $(xB)_{||} = \left(\frac{1}{n}\sum_{v\in V}(xB)(v)\right) \cdot \vec{1}$ and $(xB)_{\perp} = (xB) - (xB)_{||}$. Then, we have:

$$||xBPB||_2 \leq ||(xB)_{||}PB||_2 + ||(xB)_{\perp}PB||_2 \tag{1}$$

by triangle inequality. Here we can see that $(xB)_{||}P = (xB)_{||}$ by construction of $(xB)_{||}$. Moreover, as $(xB)_{||}$ is constant $||(xB)_{||}B||_2 = \sqrt{\beta}||(xB)_{||}||_2$. But:

$$\begin{aligned} ||(xB)_{||}||_2 = \sqrt{n\left(\frac{1}{n}\sum_{v\in V}(xB)(v)\right)^2} &= \sqrt{\frac{1}{n}(\sum_{v\in V}(xB)(v))^2} = \\ &= \frac{1}{\sqrt{n}}\sum_{v\in V}(xB)(v) \end{aligned}$$

And, by Cauchy-Schwarz we get:

$$||(_{||}(xB)||_2 = \frac{1}{\sqrt{n}}\sqrt{|B|}\sqrt{\sum_{v\in V}x^2(v)} = \sqrt{\beta}||x||_2$$

So, we obtain $||(xB)_{||}PB||_2 \leq \beta||x||_2$.

To bound the second term in the sum in Equation 1, we use the spectral gap of P. Recall that by the construction of the transition matrix and the assumption of the theorem, P has largest eigenvalue 1 and other eigenvalues $\frac{\lambda_2}{d}, \cdots, \frac{\lambda_n}{d}$, all less or equal to α in absolute value. Let $y_1, y_2, \cdots, y_n$ be an orthonormal basis of eigenvectors for P. Let z be any vector perpendicular to $\vec{1}$ and $z = \sum_{i=2}^{n} z_i y_i$. Then:

$$\begin{aligned} ||zP||_2 = ||\frac{1}{d}\sum_{i=2}^{n} z_i\lambda_i y_i||_2 &\leq \frac{1}{d}\sqrt{\sum_{i=2}^{n} z_i^2\lambda_i^2} \leq \\ &= \max_{i=2,\cdots,n}\left|\frac{\lambda_i}{d}\right|\sqrt{\sum_{i=2}^{n} z_i^2} \leq \alpha||z||_2 \end{aligned}$$

In particular, this implies $||(xB)_{\perp}PB||_2 \leq ||(xB)_{\perp}P||_2 \leq \alpha||(xB)_{\perp}||_2 \leq \alpha||x||_2$. Finally, this yields:

$$||xBPB||_2 \leq ||(xB)_{||}PB||_2 + ||(xB)_{\perp}PB||_2 \leq (\alpha+\beta)||x||_2$$

$\square$

Notice that from the same argument we could have obtained $||xBPB||_2 \leq \max(\alpha, \sqrt{\beta})||x||_2$. But in our application α can be made arbitrarily small, so that $\alpha + \beta$ is a better choice.

Relation between spectral gap and diameter of graph

In this section we prove the following theorem:

Theorem 8 *Let G=(V,E) be a d-regular graph with $\lambda(G) \leq \alpha d$. Then, the diameter of G is* $O\left(\frac{1}{1-\alpha} \log n\right)$.

PROOF: Let P be transition matrix of the random walk of G, i.e. $P(u,v) = \frac{1}{d} M_G(u,v)$. Then, the largest eigenvalue of P is 1 and all remaining eigenvalues $\lambda_2, \cdots, \lambda_n$ have absolute value less or equal to α. $\square$

Let x be a vector representing a distribution over vectors, $\forall v \in V \; x(v) \geq 0$ and $\sum_{v \in V} x(v) = 1$. As $\langle x, \pi_u \rangle = 1$, $\pi_u - x$ is perpendicular to $\vec{1}$. Hence, for any k:

$$||\pi_u - xP^k||_2 = ||(\pi_u - x)P^k||_2 \leq \alpha^k ||\pi_u - x||_2$$

Suppose x represents a probability distribution with all probability mass concentrated at one vertex. Then $||\pi_u - x||_2 = \sqrt{(n-1)\frac{1}{n^2} + \left(\frac{n-1}{n}\right)^2} \leq 2$. Hence, for $k = \log_{\frac{1}{\alpha}}(10n) = O(\frac{1}{1-\alpha}\log n)$:

$$||\pi_u - xP^k||_2 \leq 2\alpha^k \leq \frac{1}{5n}$$

which implies that for every vertex v:

$$|\pi_u(v) - xP^k(v)| \leq \frac{1}{5n}$$

and hence $xP^k(v) > 0$ for all $v \in V$, implying that the diameter of G is $O(\frac{1}{1-\alpha}\log n)$.