

Problem Set 8

Electronic submission via Gradescope due **11:59pm Tuesday 11/19**. You are strongly encouraged to submit a homework with a partner—that is, submit one homework with both of your names.

*[You may discuss these problems with classmates. Feel free to look at wikipedia, course notes, etc. for reference material, but do not try to specifically search online for solutions to the problems. **Your submission must be the original work of you and your partner, and you must understand everything that is written on your submission.** We strongly suggest that you write solutions using LaTeX—see the course website for a latex solution template.]*

1. (A Couple of Couplings)

- (a) (2 points) Consider a Markov Chain on n states, with the property that there is some state s s.t. no matter what state you are in at time t , the probability that you are in state s at time $t+1$ is at least $\alpha > 0$. (I.e. there is a column of the transition matrix whose smallest entry is at least α .) Define a coupling of this Markov chain and prove that the mixing time is at most $1 + \frac{\log(1/2e)}{\log(1-\alpha)}$. [Hint: When defining the coupling, be careful to make sure that the transition probabilities sum to 1.]
- (b) (2 points) Consider the Markov chain over 5 states, with transition matrix P defined as:

$$P = \begin{bmatrix} 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}.$$

Prove that the corresponding Markov chain has mixing time at most $2 \left(1 + \frac{\log(1/2e)}{\log(13/16)} \right)$. [Hint: feel free to use the observation that the minimum entry of the matrix P^2 is $3/16$.]

2. (4 points) Consider a random walk on the n -dimensional hypercube (i.e. where the vertices are n -bit numbers, and edges connect numbers that differ in a single index), where in each step, with probability $1/2$ one stays at the current vertex, and with probability $1/2$, one chooses a random index $i \leftarrow \{1, \dots, n\}$ and flips the i th coordinate. Prove that the mixing time is $O(n \log n)$. [Hint: When defining the coupling, it might be helpful to view the above random walk as follows: pick a random index $i \leftarrow \{1, \dots, n\}$, and a value $b \leftarrow \{0, 1\}$, and update the i th coordinate to have value b .]
3. (More Shuffling) Consider the shuffling algorithm where at each time step, a uniformly random card is chosen, and placed on the top of the deck. [We considered this shuffling algorithm in Lecture 15.] Assume the deck has n cards. We will prove that if the chain is run for $0.9 \cdot n \log n$ steps, the total variational distance to the uniform distribution is $1 - o(1)$ (hence the deck is essentially “unshuffled”), whereas if the chain is run for $1.1 \cdot n \log n$ steps then it is essentially shuffled.

- (a) (2 points) Prove that this shuffling defines an irreducible, aperiodic chain (over $n!$ states), whose stationary distribution is the uniform distribution over orderings of the n cards.
- (b) (2 points) Prove that for any initial ordering of the cards, σ , the distance between the distribution of orderings after t time steps, and the uniform distribution, $\|p_\sigma^t - \text{Unif}\|_1 = o(1)$ for $t = 1.1 \cdot n \log n$, where p_σ^t denotes the distribution of the state of the deck after t timesteps when started in the order σ . Hence the deck is essentially shuffled after $1.1n \log n$ steps. [We essentially saw this in lecture 15, but skipped the details. You can show this either via a “strong stationary stopping time” or via a coupling. . .]
- (c) (2 points) Prove that after $0.9 \cdot n \log n$ steps, with probability $1 - o(1)$ there will be roughly $c(n) = n^{0.1}$ cards that have not yet been picked. Feel free to use the approximation $(1 - 1/n)^n \approx 1/e$. [Hint: Approximate the expected number of unpicked cards, and bound the variance, then apply Chebyshev’s inequality.]
- (d) (2 points) Given the previous part, prove that the total variation distance to the uniform distribution over orderings of the deck will be $1 - o(1)$ after $0.9n \log n$ steps. [Hint: if $c(n)$ cards have not been picked, what does it mean about the ordering of the bottom $c(n)$ cards of the deck?]
4. (Azuma-Hoeffding) (4 points) Suppose you are in charge of a potato-packing plant. In a typical day, you arrive at work, and have a bin of n potatoes, which must pack into bags in such a way that each bag has at least 10 pounds of potatoes. You implement an extremely complicated potato-sorting algorithm that ensures that, each day, you *maximize* the number of bags of potatoes that are packed using the n potatoes. Suppose that the weight of each potato is between 0.5 and 2 pounds, and is drawn independently from some distribution P . Let X denote the random variable representing the number of bags of potatoes that you will produce in a given day. Prove that $\Pr[|X - \mathbf{E}[X]| > \lambda] \leq 2e^{-\frac{\lambda^2}{2n}}$. [Hint: Use Azuma-Hoeffding tail bounds! In general, if you want to make a martingale argument, but don’t know where to begin, try defining the Doob martingale....]
5. (Martingale Stopping Theorem) Consider the following very simplified model of how opinions propagate in social networks: suppose we represent the class via an undirected social network, $G = (V, E)$. The students are vertices, and an edge connects two students if they are “friends”. Assume that each student either believes in climate change (ie is a “believer”) or is a climate change “denier”. Furthermore, assume that people’s states evolve over time according to the following process: each week, each student flips a fair coin, and with probability $1/2$ keeps their current state, and with probability $1/2$ adopts the state of a randomly chosen “friend”. What can we say about the spread of this contagion?
- (a) (1 points) Let X_t be the number of “deniers” after t weeks. Is $\{X_t\}$ a martingale?
- (b) (2 points) Let Y_t be the sum of the degrees of all deniers after the t th week. Is $\{Y_t\}$ a martingale?
- (c) (2 points) Using the martingale stopping theorem, what is the probability that the entire class eventually (i.e. after an infinite number of weeks) ends up a “denier”, as a function of the starting configuration?