Problem Set 6

Electronic submission to Gradescope due **10am Thursday 11/17**. You are strongly encouraged to submit a homework with a partner—that is, submit one homework with both of your names. If you submit with a partner, you must use a different partner than in previous problem sets!

[You may discuss these problems with classmates. Please do not troll the internet looking for solutions to these problems. **Your submission must be the original work of you and your partner, and you must understand everything that is written on your submission.** We suggest that you write solutions using LaTeX—see the course website for a latex solution template.]

1. Consider a set of equations over variables $x_1, \ldots, x_n$, with each equation has the form $a_1x_{i_1} + a_2x_{i_2} + \ldots + a_r x_{i_r} \equiv a_{r+1} \mod 17$, for some $r$ (that might vary from equation to equation) and set of coefficients $a_1, \ldots, a_r \in \{1, 2, \ldots, 16\}$, and $a_{r+1} \in \{0, \ldots, 16\}$. Additionally, suppose that each variable, $x_i$, occurs in at most 4 equations.
   
   (a) Prove that there exists and assignment to the variables such the none of the equations are satisfied. [Hint: Lovasz Local Lemma!]
   
   (b) Would your proof above continue to hold if the equations were modulo 18 rather than modulo 17?

2. For a parameter $p \in [0, 1]$, the **binary symmetric Markov chain** with parameter $p$ is a two state Markov chain, with states 0 and 1, and transition probabilities defined as follows: the probability of staying in the current state is $p$, and the probability of switching states is $1 - p$. Prove that if one starts in state 0 at time 0, the probability that one is in state 0 at time $t$ is $(2p - 1)^t + 1$ (and one is in state 1 at time $t$ with probability $1 - (2p - 1)^t$).

3. Consider a social network with $n$ people, with (undirected) edges connecting friends. Assume the network is connected. At time $t = 1$, suppose one person is "sick". For all $t \geq 1$, assume that anyone who is sick at time $t - 1$ becomes "healthy" with probability 1/2 (and stays "sick" with probability 1/2). Additionally, any "healthy" person will become sick with probability 1/2 if he/she has at least 1 “sick” friend (and otherwise, remains healthy). Note that the above setup describes a Markov chain, where each state of the chain is a configuration of “healthy”/"sick" people. Answer each question with one sentence of explanation:
   
   (a) What is an upper bound on the size of the state space of the Markov chain?
   
   (b) Is the chain irreducible?
   
   (c) In the limit as $t \to \infty$, what will the state be at time $t$?

4. For each of the following Markov processes, state whether it is **irreducible**, and whether it is **aperiodic**. No need to give proofs.
   
   (a) The chain defined by a random walk on a (connected) bipartite graph. (Namely the states of the chain are the nodes of the graph, and at each time step, one moves to a randomly chosen neighbor of one’s current state/node.)
(b) The chain defined by a random walk on a (connected) bipartite graph, that has been modified by adding a single edge connecting a pair of nodes on the same side.

5. Consider a finite state Markov chain defined by transition matrix $P$ (where $P_{i,j}$ is the probability of transitioning from state $i$ to state $j$). Recall the fundamental theorem of Markov chains that states that if the chain is finite, irreducible, and aperiodic, then there is a unique stationary distribution, denoted by the row-vector $\pi$, that satisfies $\pi P = \pi$.

(a) A doubly stochastic matrix $P$ is one for which all rows and all columns sum to 1. Prove that if the Markov process defined by a doubly stochastic matrix $P$ is finite, irreducible, and aperiodic, then the stationary distribution is the uniform distribution.