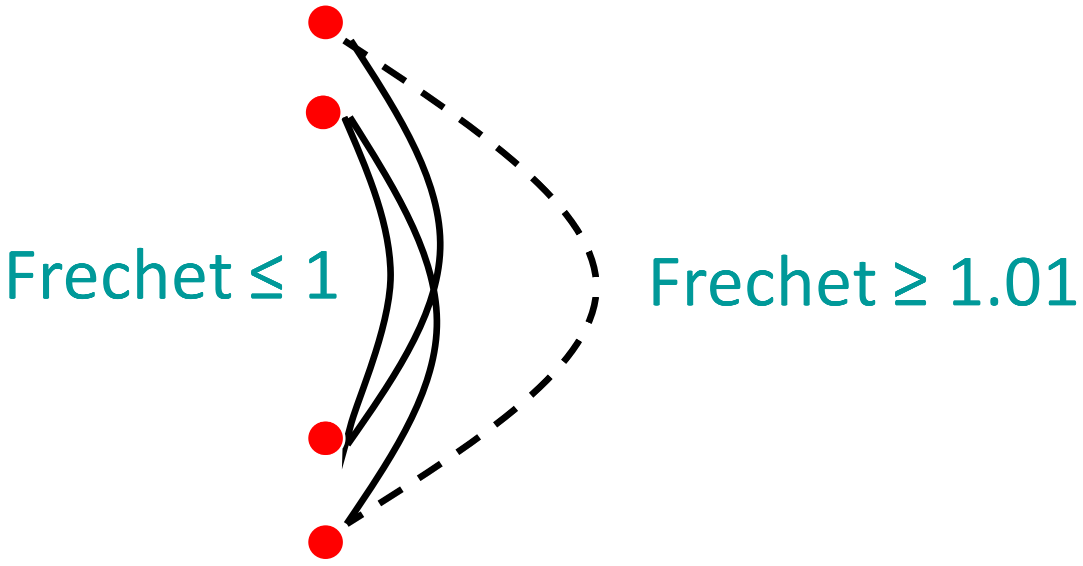
Overview of the talk

- Hardness for Frechet distance
- Hardness for edit distance
- Hardness for LCS

Coordinate gadgets

- If $a^i \cdot b^i = 1$ then Frechet ≥ 1.01
- If $a^i \cdot b^i = 0$ then Frechet ≤ 1
- $a^i = 1$
- $a^i = 0$



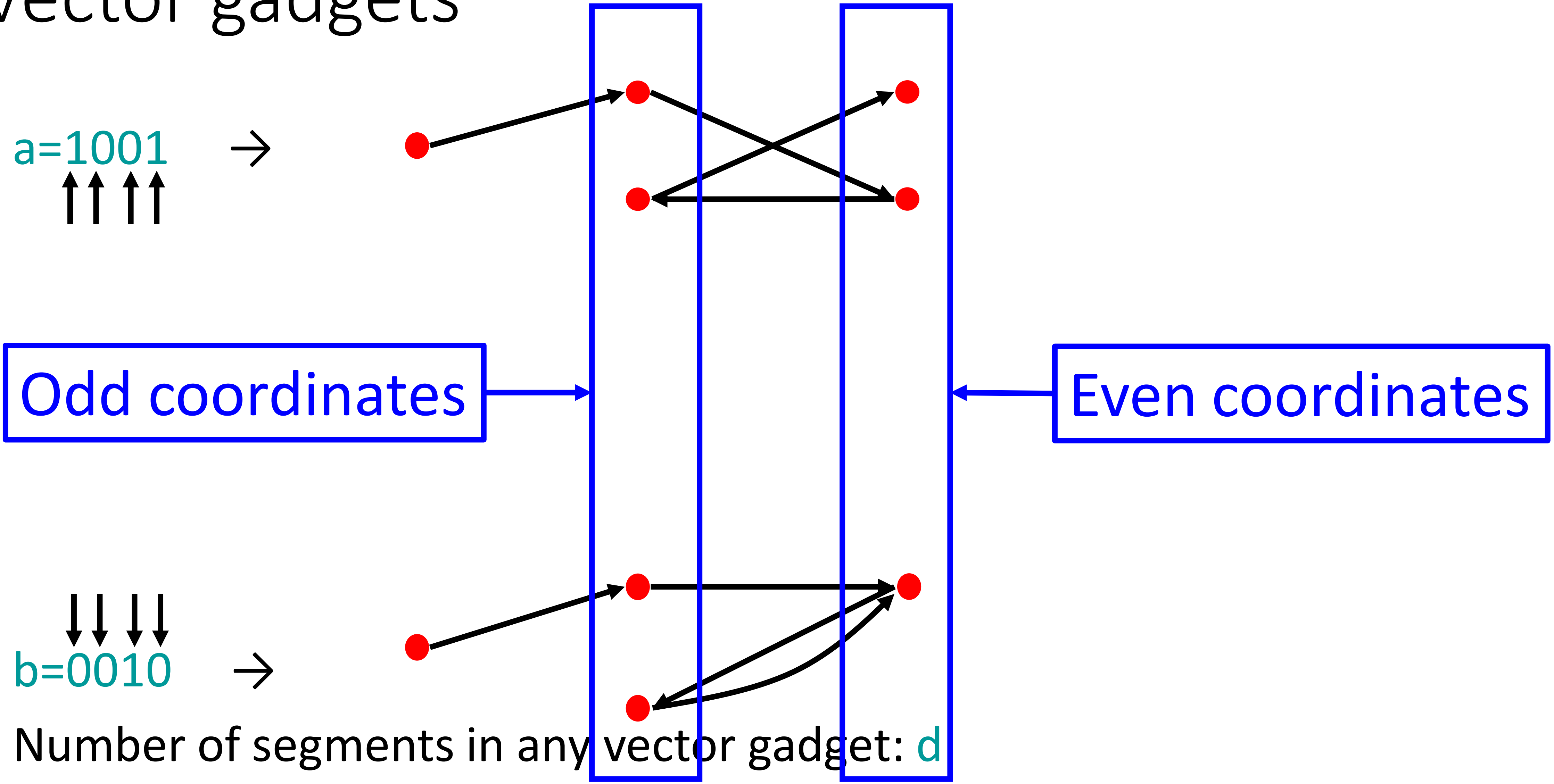
- $b^i = 0$
- $b^i = 1$

Vector gadgets

- $a_i \in A \rightarrow \alpha_i$ $|\alpha_i| \leq O(d)$
- $b_j \in B \rightarrow \beta_j$ $|\beta_j| \leq O(d)$
- If a_i and b_j are orthogonal, then $\text{Frechet}(\alpha_i,\beta_j) \leq 1$
- If a_i and b_j are NOT orthogonal, then $\text{Frechet}(\alpha_i,\beta_j) \geq 1.01$

Vector gadgets

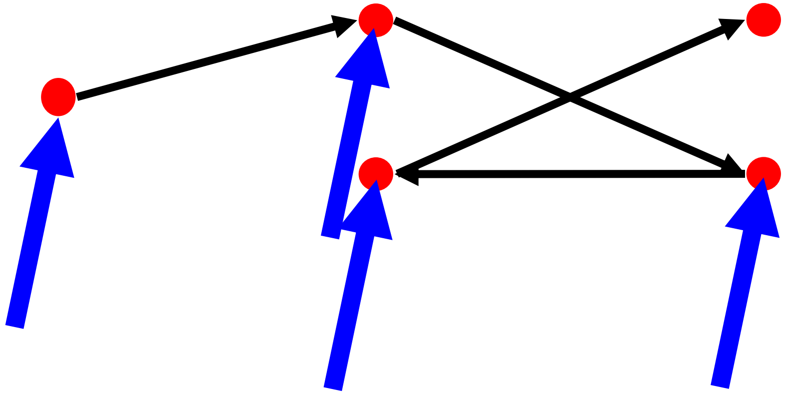
- a=1001 →



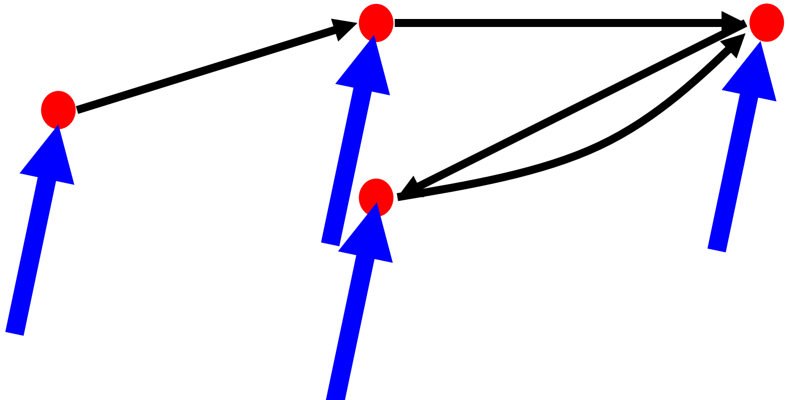
- b=0010 →
- Number of segments in any vector gadget: d

Vectors are orthogonal → Frechet ≤ 1

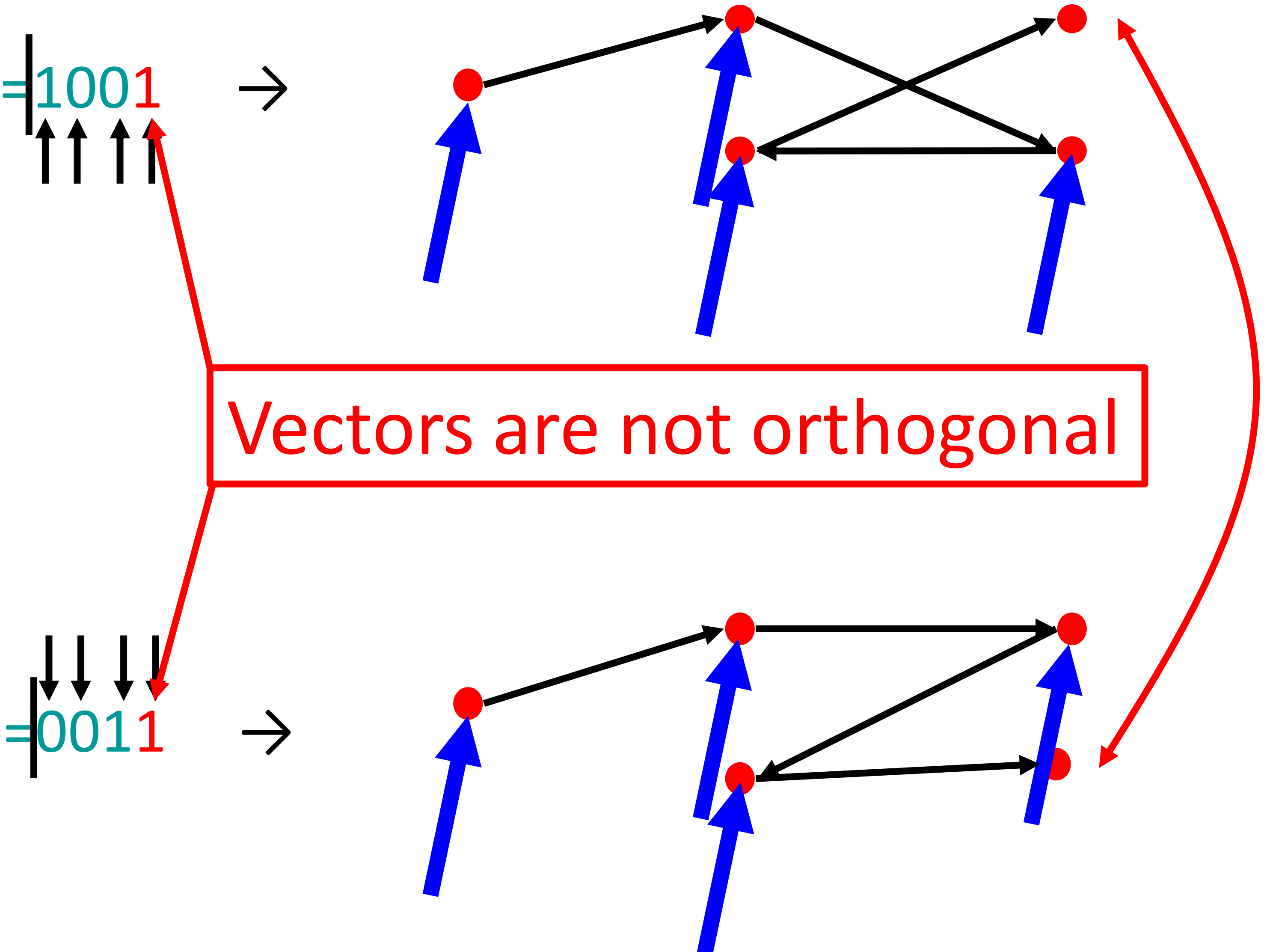
- a=1001 →



- b=0010 →



Vectors are not orthogonal → Frechet ≥ 1.01



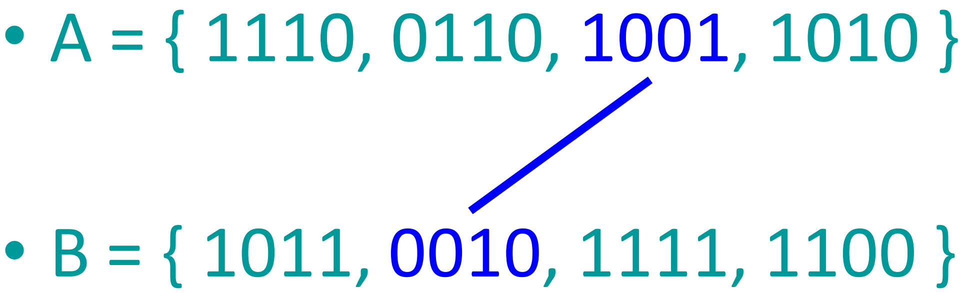
- a=1001 →

Vectors are not orthogonal

Frechet ≥ 1.01

- b=0011 →

Construction of the final curves

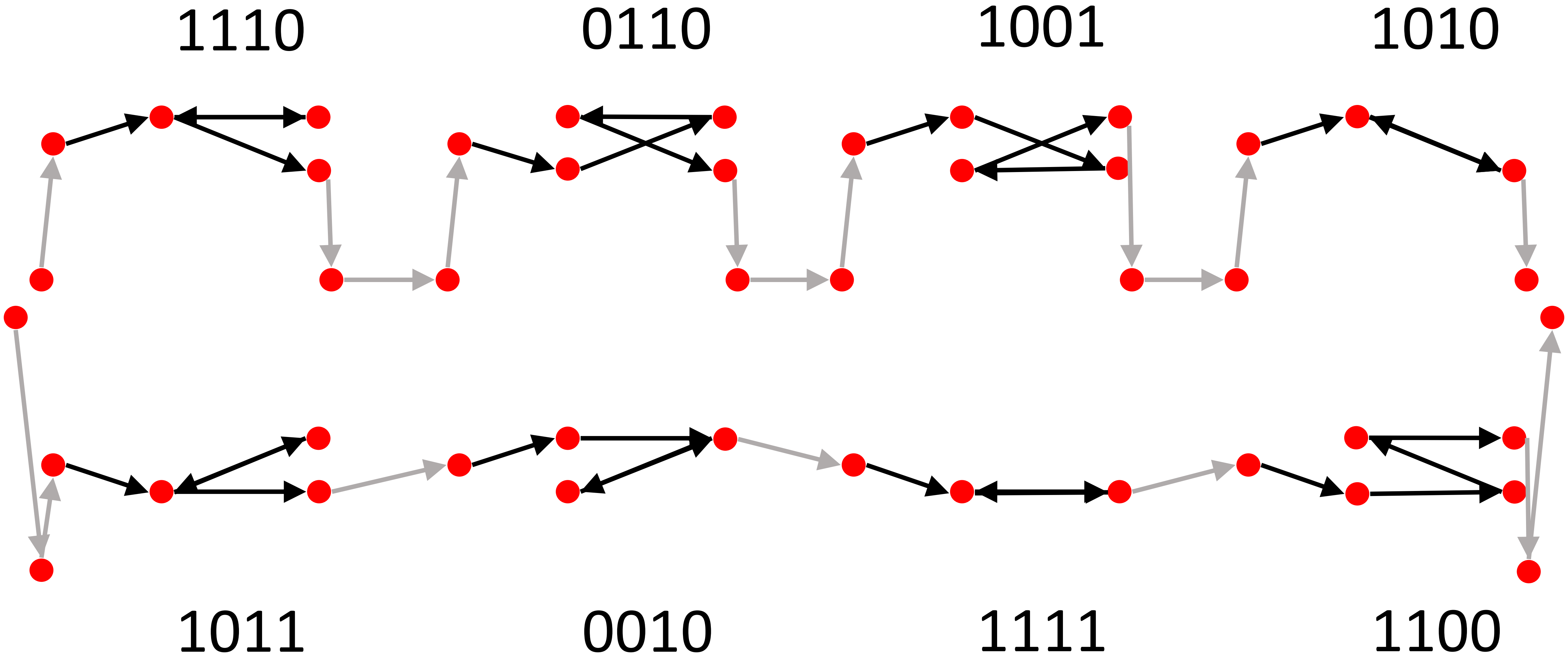


- A = { 1110, 0110, 1001, 1010 }

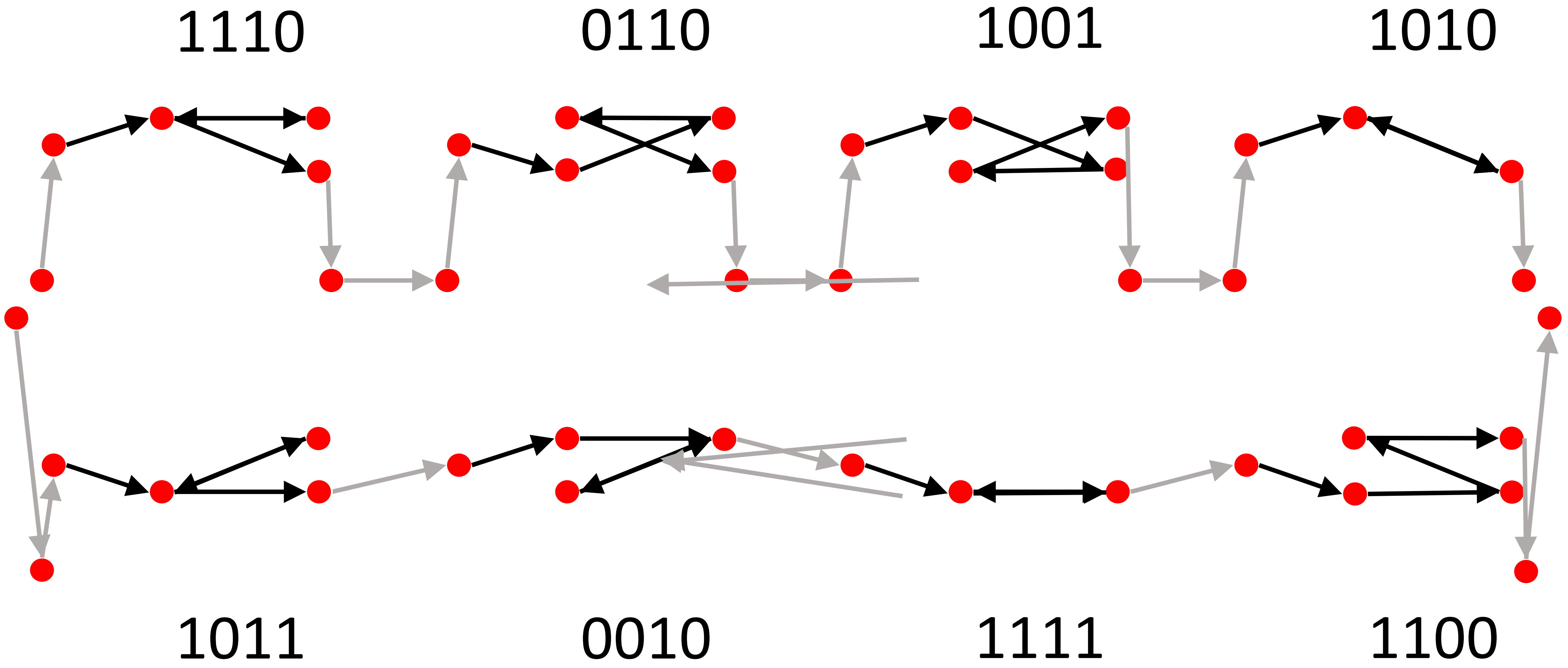
- B = { 1011, 0010, 1111, 1100 }

- One pair of orthogonal vectors

Construction of the final curves



Construction of the final curves

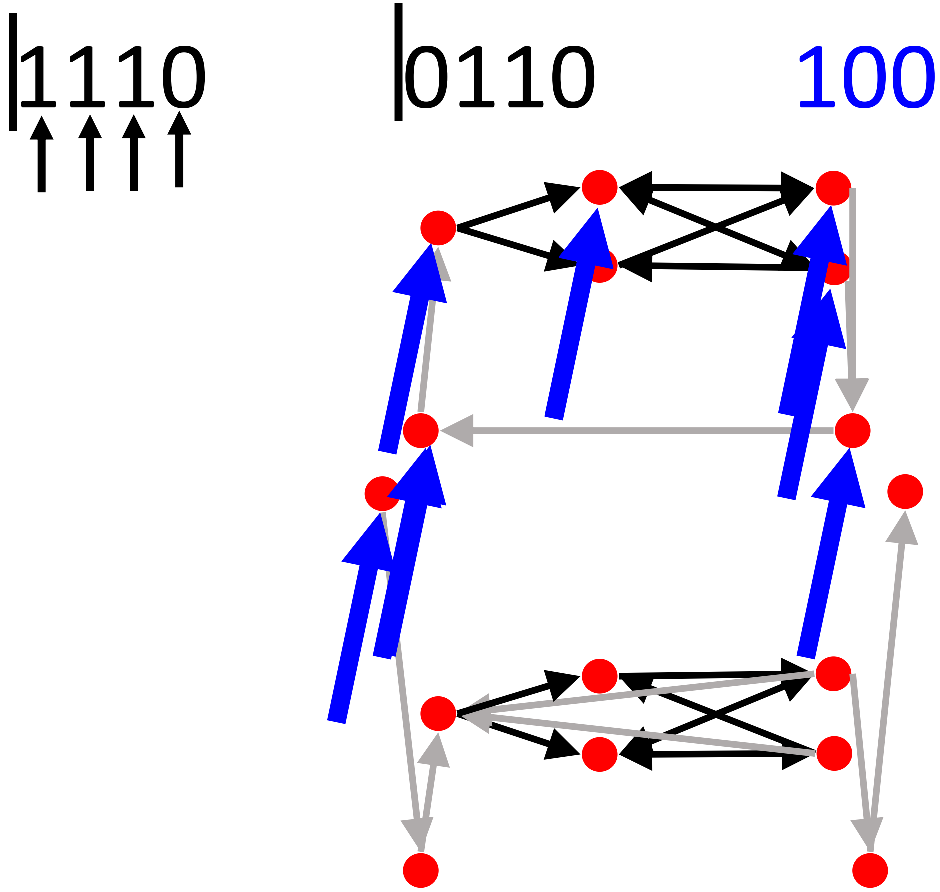


Reminder of our goal

- $A \subseteq \{0,1\}^d$ → curve x, $|x| \leq O(n \cdot d)$
- $B \subseteq \{0,1\}^d$ → curve y, $|y| \leq O(n \cdot d)$
- $\mathrm{Frechet}(x,y) \leq 1$, if exists $a \in A$, $b \in B$ with $\Sigma_i a^i b^i = 0$
- $\mathrm{Frechet}(x,y) \geq 1.01$, otherwise

Exist orthogonal vectors → Frechet(x,y) ≤ 1

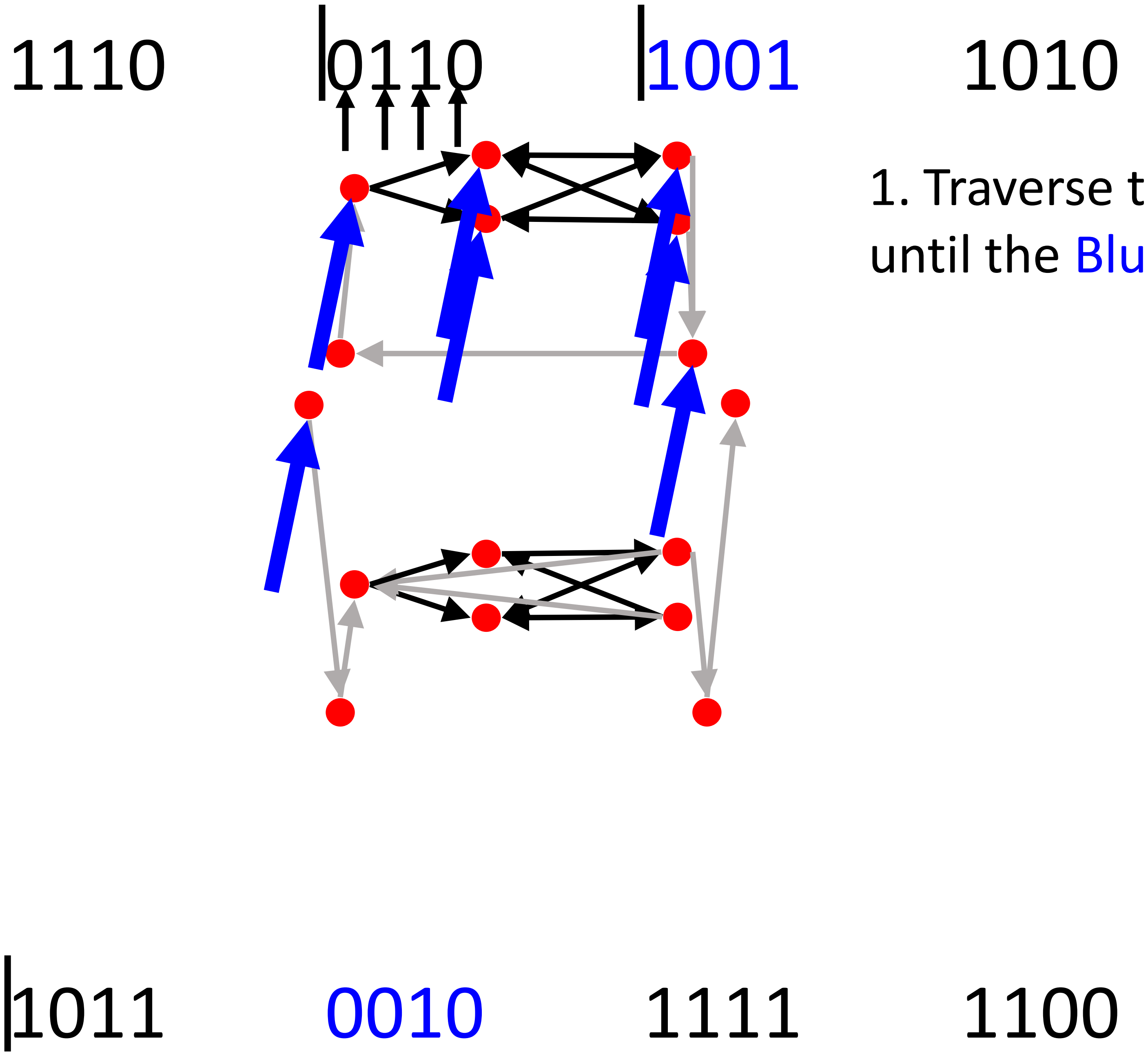
1110 0110 1001 1010



1. Traverse the **first** curve until the Blue vector

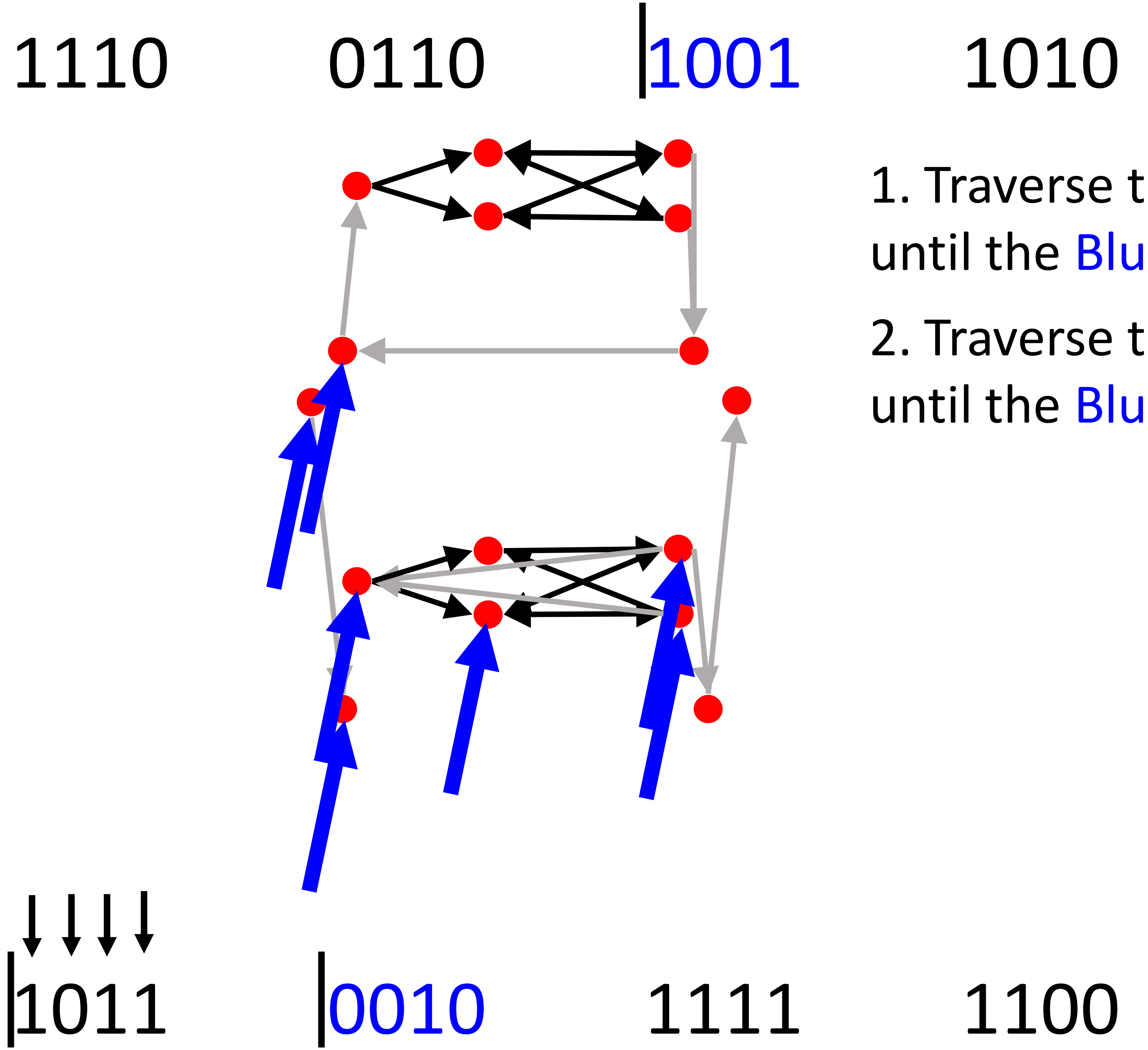
1011 0010 1111 1100

Exist orthogonal vectors → Frechet(x,y) ≤ 1



1. Traverse the **first** curve until the Blue vector

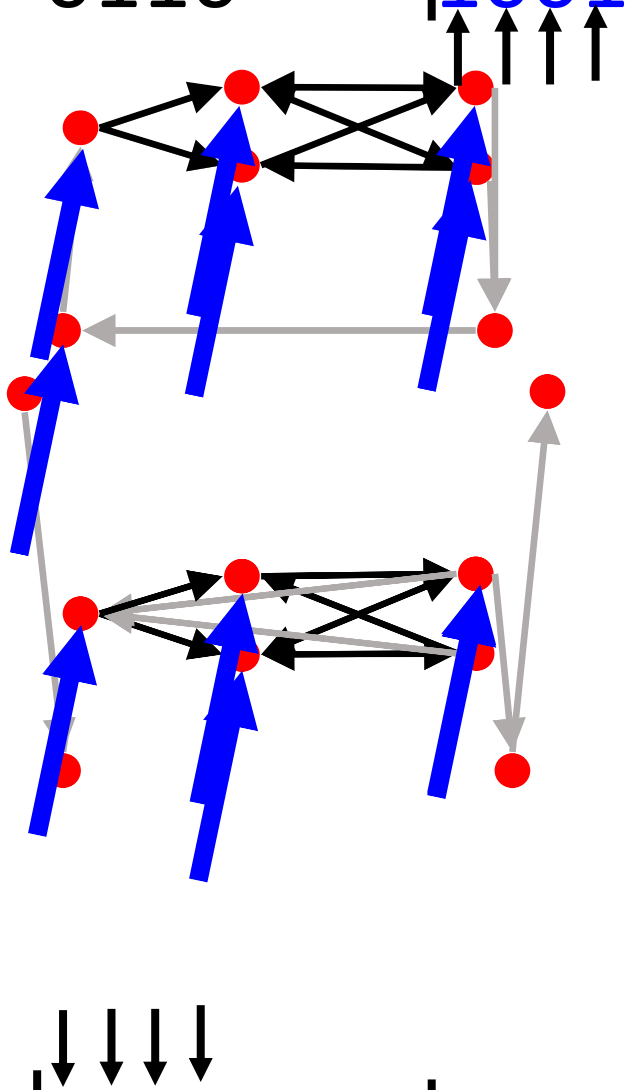
Exist orthogonal vectors → Frechet(x,y) ≤ 1



1. Traverse the **first** curve until the Blue vector
2. Traverse the **second** curve until the Blue vector

Exist orthogonal vectors → Frechet(x,y) ≤ 1

1110 0110 |1001 |1010

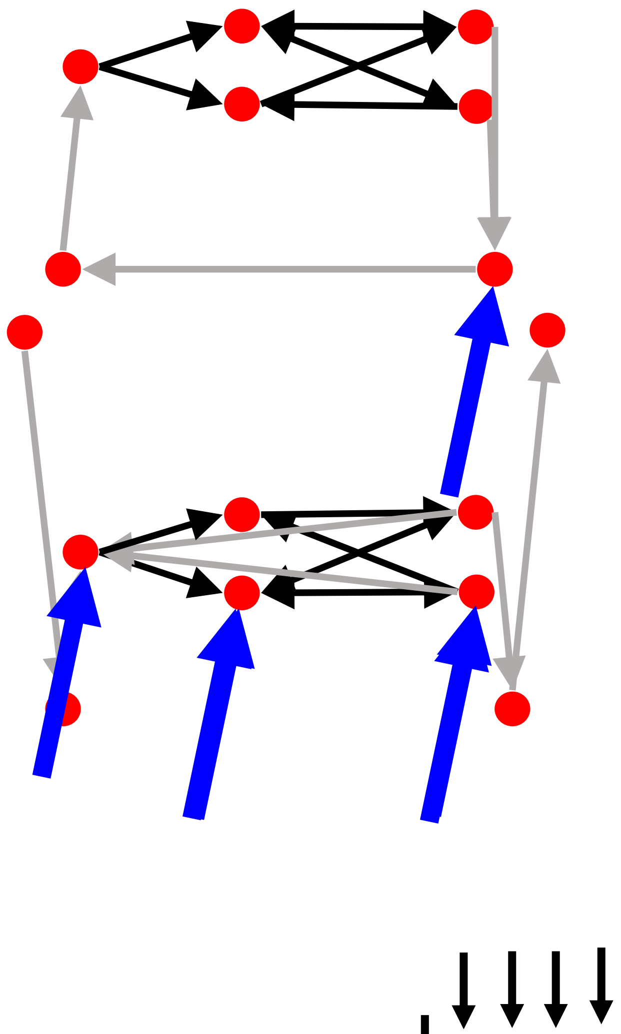


1011 |0010 |1111 1100

1. Traverse the **first** curve until the Blue vector
2. Traverse the **second** curve until the Blue vector
3. Traverse the Blue vectors in parallel
4. Traverse the **second** curve until end

Exist orthogonal vectors → Frechet(x,y) ≤ 1

1110 0110 1001 |1010

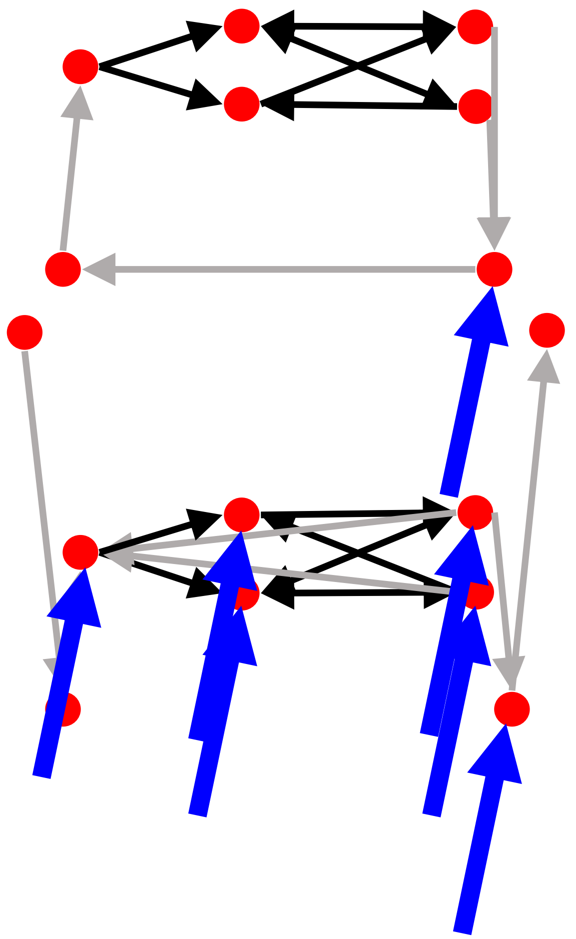


1011 0010 |1111 |1100

1. Traverse the **first** curve until the Blue vector
2. Traverse the **second** curve until the Blue vector
3. Traverse the Blue vectors in parallel
4. Traverse the **second** curve until end

Exist orthogonal vectors → Frechet(x,y) ≤ 1

1110 0110 1001 1010



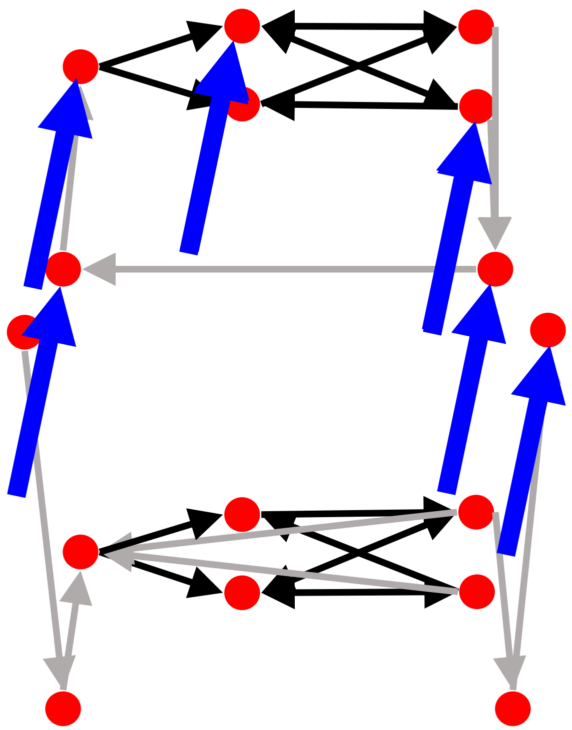
1. Traverse the **first** curve until the Blue vector
2. Traverse the **second** curve until the Blue vector
3. Traverse the Blue vectors in parallel
4. Traverse the **second** curve until end

1011 0010 1111 1100

Exist orthogonal vectors → Frechet(x,y) ≤ 1

1110 0110 1001 1010

Questions?



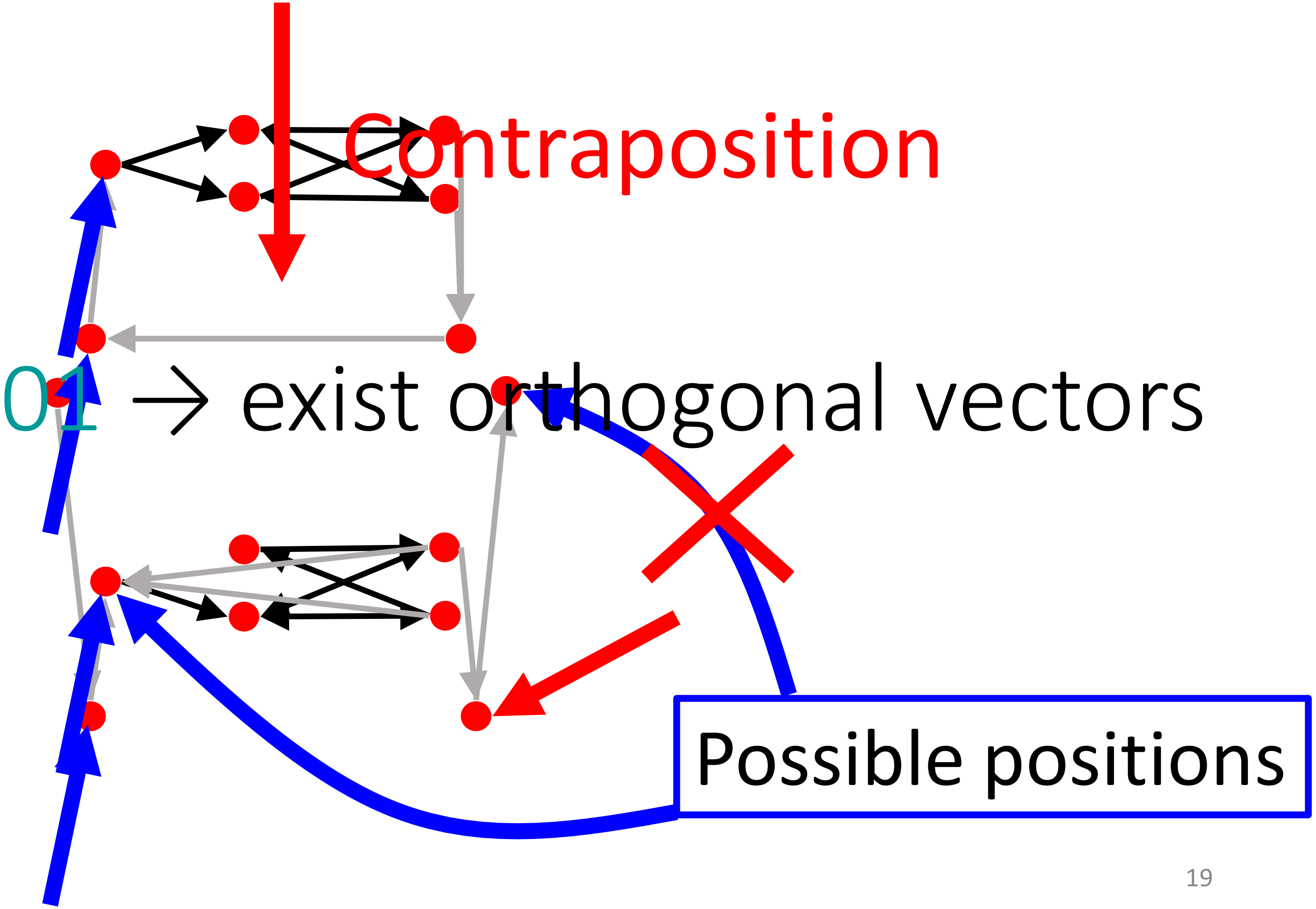
1. Traverse the **first** curve until the Blue vector
2. Traverse the **second** curve until the Blue vector
3. Traverse the Blue vectors in parallel
4. Traverse the **second** curve until end
5. Traverse the **first** curve until end

1011 0010 1111 1100

Reminder of our goal

- $A \subseteq \{0,1\}^d$ → curve x, $|x| \leq O(n \cdot d)$
- $B \subseteq \{0,1\}^d$ → curve y, $|y| \leq O(n \cdot d)$
- $\text{Frechet}(x,y) \leq 1$, if exist $a \in A$, $b \in B$ with $\Sigma_i a^i b^i = 0$
- $\text{Frechet}(x,y) \geq 1.01$, otherwise

No orthogonal vectors → Frechet(x,y) ≥ 1.01



Contraposition

Questions?

Frechet(x,y) < 1.01 → exist orthogonal vectors

Possible positions

Hardness for Frechet distance

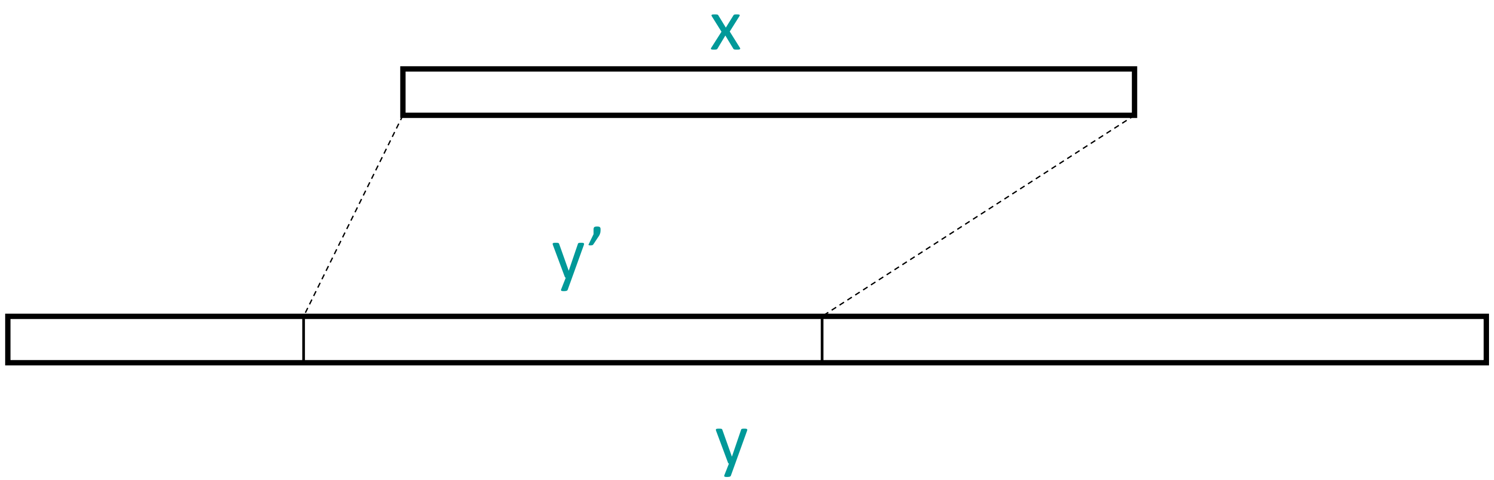
- Theorem. If Frechet distance can be 1.001-approximated in $m^{2-\Omega(1)}$ time (m upper-bounds the length of curves), then Orthogonal Vectors Conjecture is false

Overview of the talk

- Hardness for Frechet distance
- Hardness for edit distance
- Hardness for LCS

Pattern matching

- Pattern matching with respect to edit distance
- pattern(x,y), $|x| \le |y| = m$



- $\text{pattern}(x,y) = \min_{y'} \text{edit}(x,y')$

High level idea

- $A \subseteq \{0,1\}^d$ → sequence x, $|x| \leq n \cdot d^{O(1)}$
- $B \subseteq \{0,1\}^d$ → sequence y, $|y| \leq n \cdot d^{O(1)}$
- pattern(x,y)=small, if exists $a \in A$, $b \in B$ with $\Sigma_i a^i b^i = 0$
- pattern(x,y)=large, otherwise
- the construction time is $n \cdot d^{O(1)}$
- Theorem. If pattern matching can be computed in $m^{2-\Omega(1)}$ time (m upper-bounds the length of sequences), then Orthogonal Vectors Conjecture is false

Hardness for pattern matching distance

- First part: assume vector gadgets and prove hardness for pattern matching
- Second part: show vector gadget construction

Vector gadgets

- $a_i \in A \rightarrow \alpha_i$ $|\alpha_i| \leq d^{O(1)}$
- $b_j \in B \rightarrow \beta_j$ $|\beta_j| \leq d^{O(1)}$
- If a_i and b_j are orthogonal, then $\mathrm{edit}(\alpha_i,\beta_j)=S$
- If a_i and b_j are NOT orthogonal, then $\mathrm{edit}(\alpha_i,\beta_j)=L$, $S<L$
- Crucial that L does not depend on vectors

Construction of final sequences

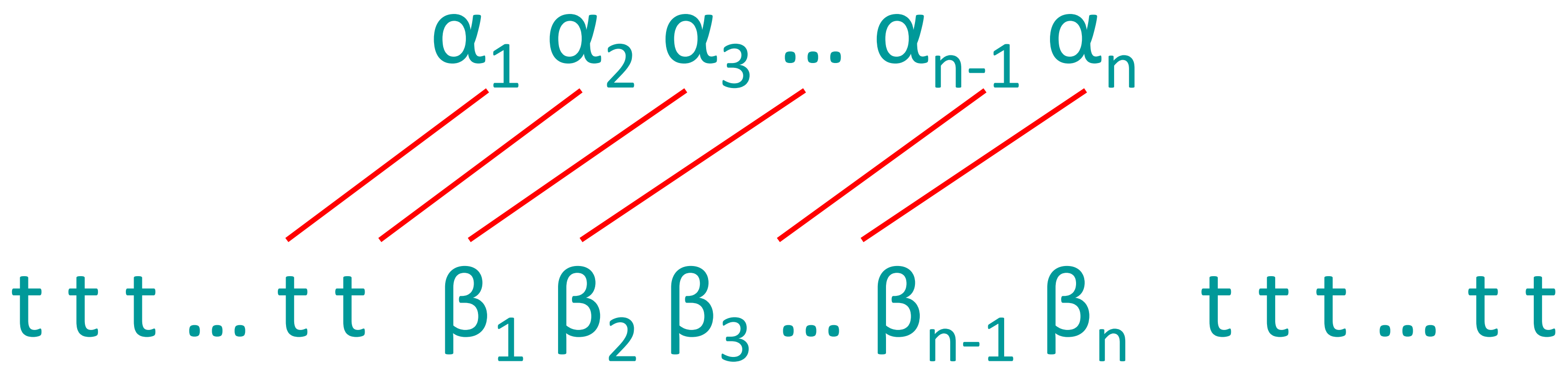
- t – vector gadget for vector consisting of 1s

$$\alpha_1 \overset{d^{O(1)}}{\$} \alpha_2 \$ \alpha_3 \dots \alpha_{n-1} \$ \alpha_n$$

$$\underbrace{t \$ t \$ t \dots t \$ t}_{n\text{-}1 \text{ times}} \$ \beta_1 \$ \beta_2 \$ \beta_3 \dots \beta_{n-1} \$ \beta_n \$ \underbrace{t \$ t \$ t \dots t \$ t}_{n\text{-}1 \text{ times}}$$

Pattern matching score

- YES case: exists $a \in A$, exists $b \in B$ such that $\Sigma_i a^i b^i = 0$
  assume that a_3 and b_1 are orthogonal

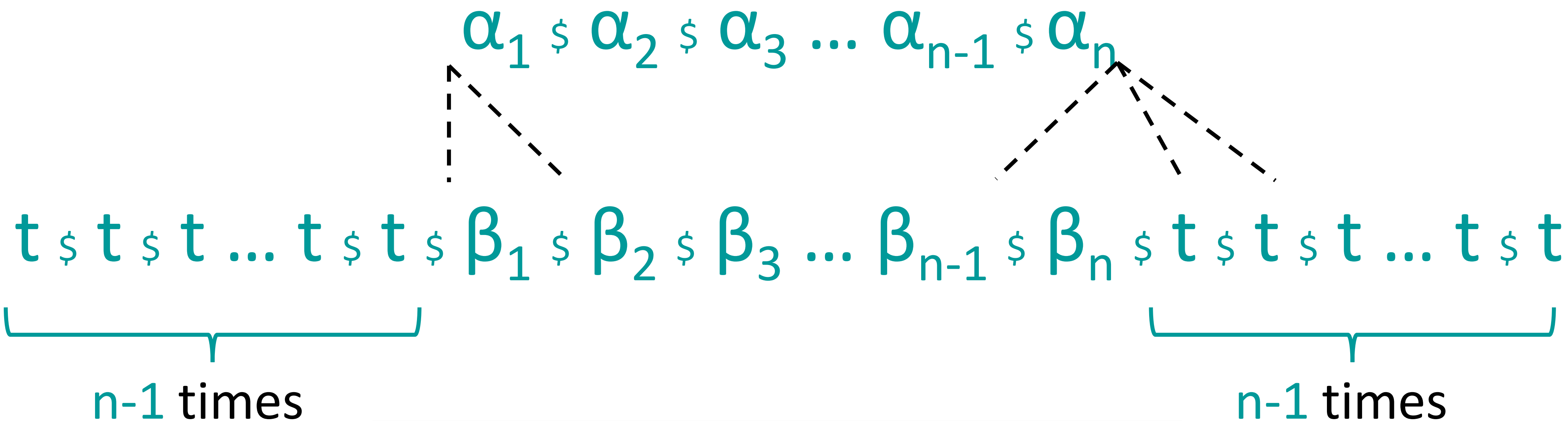


$\alpha_1\ \alpha_2\ \alpha_3\ \ldots\ \alpha_{n-1}\ \alpha_n$

$t\ t\ t\ \ldots\ t\ t\quad \beta_1\ \beta_2\ \beta_3\ \ldots\ \beta_{n-1}\ \beta_n\quad t\ t\ t\ \ldots\ t\ t$

Score: $S + (S \text{ or } L) + (S \text{ or } L) + (S \text{ or } L) \ldots + (S \text{ or } L) + (S \text{ or } L) \quad \leq S + (n-1) \cdot L$

Construction of final sequences



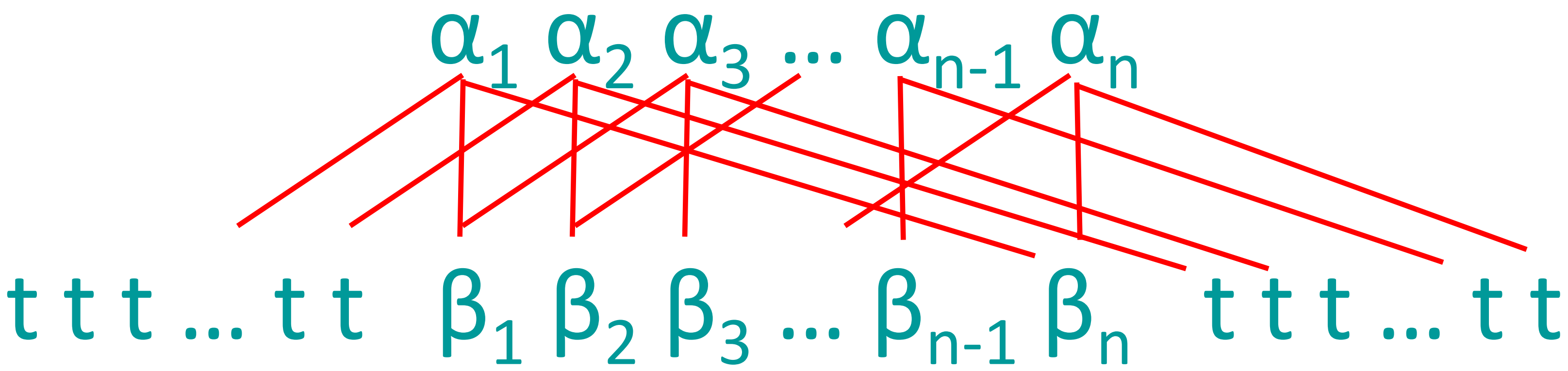
$$\alpha_1 \, \$ \, \alpha_2 \, \$ \, \alpha_3 \ldots \alpha_{n-1} \, \$ \, \alpha_n$$

$$\underbrace{t \, \$ \, t \, \$ \, t \ldots t \, \$ \, t}_{n-1 \text{ times}} \, \$ \, \beta_1 \, \$ \, \beta_2 \, \$ \, \beta_3 \ldots \beta_{n-1} \, \$ \, \beta_n \, \$ \, \underbrace{t \, \$ \, t \, \$ \, t \ldots t \, \$ \, t}_{n-1 \text{ times}}$$

Claim 2: vector gadgets are aligned one by one between the upper sequence and the subsequence

Pattern matching score

- NO case: for all $a \in A$, $b \in B$ we have $\Sigma_i a^i b^i > 0$



$\alpha_1\ \alpha_2\ \alpha_3 \dots \alpha_{n-1}\ \alpha_n$

$t\ t\ t \dots t\ t\quad \beta_1\ \beta_2\ \beta_3 \dots \beta_{n-1}\ \beta_n\quad t\ t\ t \dots t\ t$

- Pattern matching score=$n \cdot L$ $\quad(>S+(n-1) \cdot L)$
- We crucially use that L does not depend on $\Sigma_i a^i b^i$

Hardness for edit distance

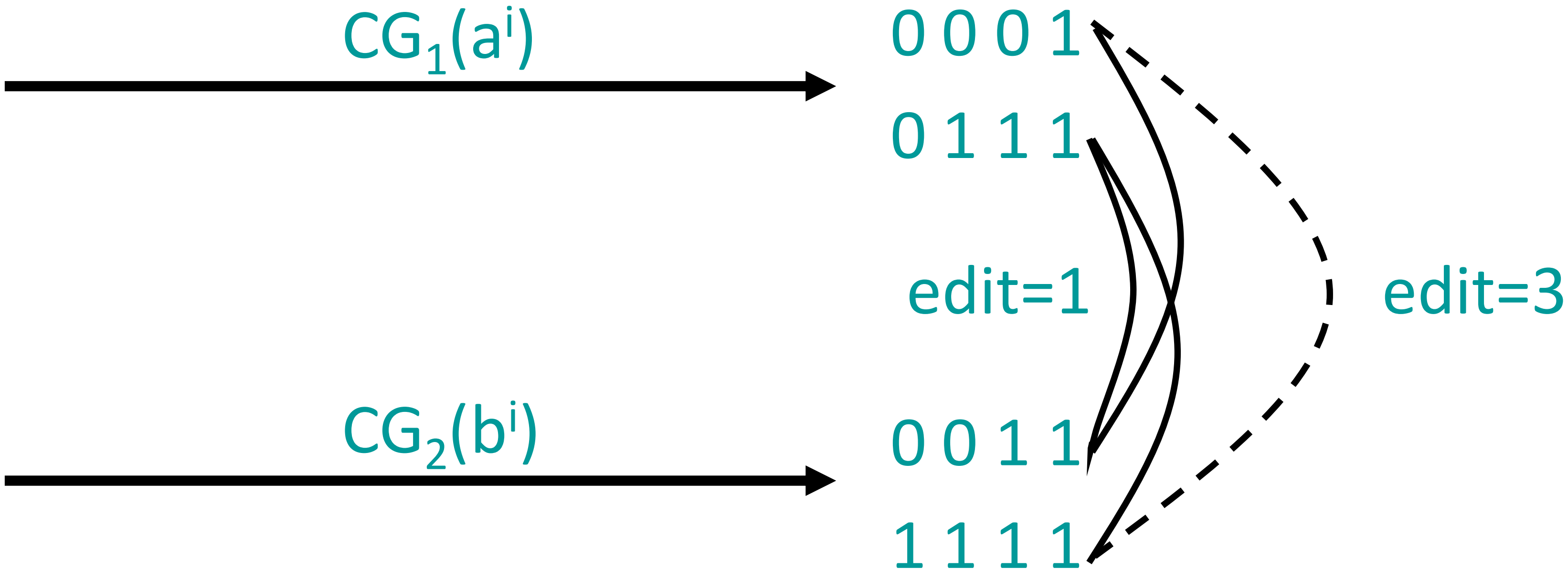
- First part: assume vector gadgets and prove hardness for pattern matching
- Second part: show vector gadget construction

Vector gadgets

- $a \in A \rightarrow \alpha$ $|\alpha| \leq d^{O(1)}$
- $b \in B \rightarrow \beta$ $|\beta| \leq d^{O(1)}$
- If $\Sigma_i a^i b^i = 0$ then $\text{edit}(\alpha,\beta) = S$
- If $\Sigma_i a^i b^i > 0$ then $\text{edit}(\alpha,\beta) = L,$ $S < L$

Coordinate gadgets

- If $a^i \cdot b^i=1$ then edit=large
- If $a^i \cdot b^i=0$ then edit=small
- $a^i=1$
- $a^i=0$
- $b^i=0$
- $b^i=1$



$CG_1(a^i)$

0 0 0 1

0 1 1 1

$CG_2(b^i)$

0 0 1 1

1 1 1 1

edit=1

edit=3

Vector gadgets: first attempt

$\alpha' =$

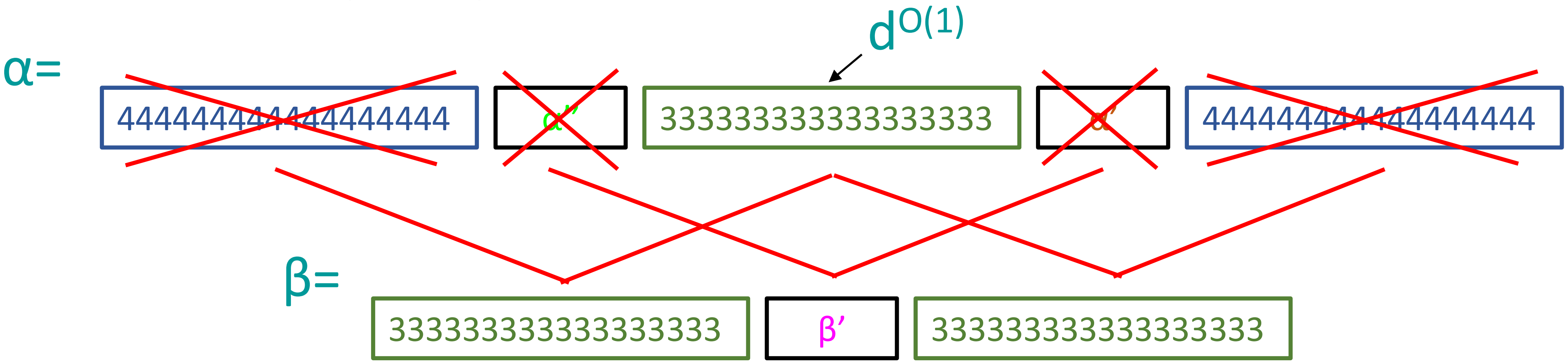
- $a=a^1a^2\ldots a^d \rightarrow CG_1(a^1)\ \overset{d^{O(1)}}{22\ldots2}\ CG_1(a^2)\ 22\ldots2\ldots\ldots22\ldots2\ CG_1(a^d)$

$\beta' =$

- $b=b^1b^2\ldots b^d \rightarrow CG_2(b^1)\ 22\ldots2\ CG_2(b^2)\ 22\ldots2\ldots\ldots22\ldots2\ CG_2(b^d)$

- $\text{edit}(\alpha',\beta')=d+2\cdot(\Sigma_i a^i b^i)$
- $\text{edit}(\alpha',\beta')$ depends on the inner product
- Pattern matching for Hamming distance is easy

Vector gadgets: final construction



$d^{O(1)}$

α=

444444444444444444

~~α’~~

333333333333333333

~~α’~~

444444444444444444

β=

333333333333333333

β’

333333333333333333

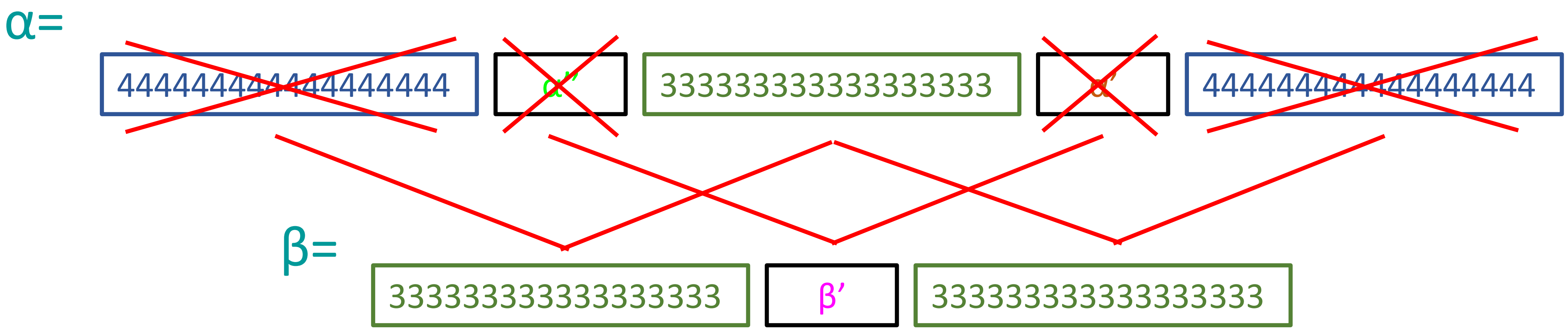
- α’= $CG_1(a^1)$ 22…2 $CG_1(a^2)$ 22…2……22…2 $CG_1(a^d)$
- β’= $CG_2(b^1)$ 22…2 $CG_2(b^2)$ 22…2……22…2 $CG_2(b^d)$
- α’’= $CG_1(0)$ 52…2 $CG_1(0)$ 22…2……22…2 $CG_1(0)$
- edit(α’,β’)=$d+2\cdot(\Sigma_i a^i b^i)$
- edit(α,β)≈$\min(d+2\cdot(\Sigma_i a^i b^i),d+1)$

Properties of vector gadgets

- $a=a^1a^2\ldots a^d \rightarrow \alpha' = CG_1(a^1)\ 22\ldots2\ CG_1(a^2)\ 22\ldots2\ldots\ldots22\ldots2\ CG_1(a^d)$
- $b=b^1b^2\ldots b^d \rightarrow \beta' = CG_2(b^1)\ 22\ldots2\ CG_2(b^2)\ 22\ldots2\ldots\ldots22\ldots2\ CG_2(b^d)$
- $\alpha'' = CG_1(0)\ 52\ldots2\ CG_1(0)\ 22\ldots2\ldots\ldots22\ldots2\ CG_1(0)$

- $\text{edit}(\alpha'', \beta') = \text{edit}(\ CG_1(0)\ 52\ldots2\ CG_1(0)\ 22\ldots2\ldots\ldots22\ldots2\ CG_1(0)$ $CG_2(b^1)\ 22\ldots2\ CG_2(b^2)\ 22\ldots2\ldots\ldots22\ldots2\ CG_2(b^d)\ ) = d+1$
- $\text{edit}(\alpha', \beta') = d$ if $\Sigma_i a^i b^i = 0$
- $\text{edit}(\alpha', \beta') = d + 2\Sigma_i a^i b^i \geq d+2$ if $\Sigma_i a^i b^i > 0$

Vector gadgets: final construction



α=

~~44444444444444444~~ ~~α''~~ 33333333333333333 ~~α'~~ ~~44444444444444444~~

β=

33333333333333333 β' 33333333333333333

edit(α,β) -C

=min (edit(α',β') +replace 3333333333 with 4444444444 +delete α'' +delete 4444444444 , edit(α'',β') +replace 3333333333 with 4444444444 +delete α' +delete 4444444444)

Vector gadgets: final construction

- $\text{edit}(\alpha,\beta)=\min(\text{edit}(\alpha',\beta'), \text{edit}(\alpha'',\beta')) + C$

- $\text{edit}(\alpha', \beta')=d$ if the vectors are orthogonal
- $\text{edit}(\alpha', \beta')\geq d+2$ if the vectors are NOT orthogonal

- $\text{edit}(\alpha'', \beta')=d+1$ always

- $\text{edit}(\alpha,\beta)=d + C$ if the vectors are orthogonal
- $\text{edit}(\alpha,\beta)=d+1 + C$ if the vectors are NOT orthogonal

Hardness for edit distance

- Theorem. If edit distance can be computed in $m^{2-\Omega(1)}$ time (m upper-bounds the length of sequences), then Orthogonal Vectors Conjecture is false

- Gap is too small to obtain interesting approximation hardness result

Overview of the talk

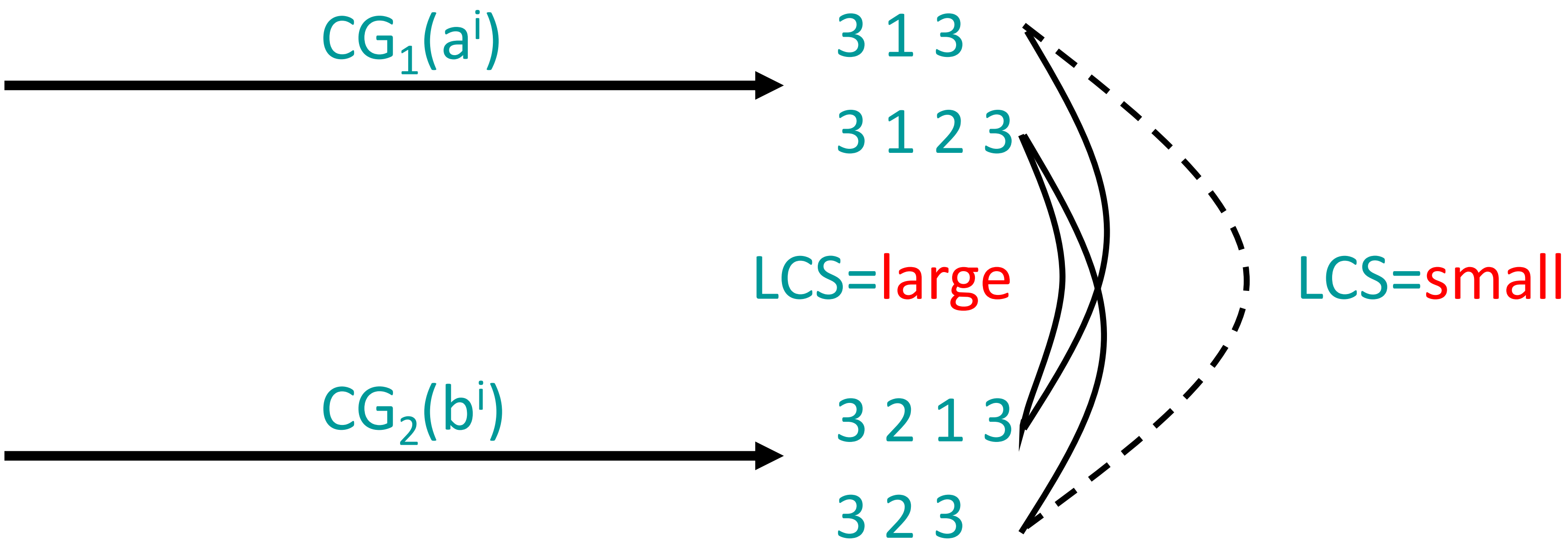
- Hardness for Frechet distance
- Hardness for edit distance
- Hardness for LCS

Hardness for LCS

- Hardness for weighted LCS
- Reduction to unweighted LCS: write symbols in unary
- Alphabet 1,2,3,4,5,6,7
- 1=Weight(1)=Weight(2)
- 1<<Weight(3)<<Weight(4)<<Weight(5)<<Weight(6)=Weight(7)

Coordinate gadgets

- If $a^i \cdot b^i=1$ then LCS=small
- If $a^i \cdot b^i=0$ then LCS=large
- $a^i=1$
- $a^i=0$
- $b^i=0$
- $b^i=1$



$CG_1(a^i)$

3 1 3

3 1 2 3

LCS=large

LCS=small

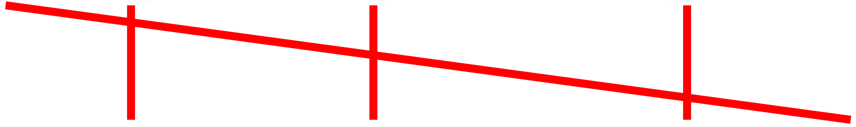
$CG_2(b^i)$

3 2 1 3

3 2 3

Vector gadgets

- $a=a^1a^2\ldots a^d \rightarrow \alpha=4\ CG_1(a^1)\ CG_1(a^2)\ \ldots\ CG_1(a^d)$



- $b=b^1b^2\ldots b^d \rightarrow \beta=\ \ CG_2(b^1)\ CG_2(b^2)\ \ldots\ CG_2(b^d)\ 4$

- $LCS(\alpha,\beta)=$large if orthogonal
- $LCS(\alpha,\beta)=$small if NOT orthogonal

Construction of final sequences

- t – vector gadget for vector consisting of 1s

Pair of orthogonal vectors → LCS is large

t t t β β β β t t t

No pair of orthogonal vectors → LCS is small

n-1 times n-1 times

Hardness for LCS

- Theorem. If LCS can be computed in $m^{2-\Omega(1)}$ time (m upper-bounds the length of sequences), then Orthogonal Vectors Conjecture is false

Conclusion

- Quadratic hardness for Frechet, edit, LCS
- Quadratic hardness for DTWD (reduce edit to DTWD)
- Open problems: hardness of approximations for edit, LCS

Thank you!