# Facility Location with Interference

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**Abstract**

We consider a variant of the facility location problem with additional interference constraints of the form "not too many facilities can be close to a user". These constraints arise naturally in the placement of wireless access points or base stations. We model the interference in several (increasingly complex) ways, and show approximation algorithms for them. We also consider the assignment of frequencies to the facilities with the motivation that two facilities with different frequencies do not interfere. This leads to an interesting coloring variant of facility location. Our techniques show that interference constraints are not too hard to enforce in rounding schemes that preserve locality.

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# 1 Introduction

In this paper we consider a generalization of the facility location problem where we want to place facilities in order to cover a certain set of client nodes. Additionally, open facilities that are close interfere with each other. Hence, for each open facility $i$ we restrict the number of other facilities in a certain neighborhood of $i$.

The problem is partly motivated by an application in wireless communication systems where one wishes to place wireless access points such that a prespecified area is covered. The wireless base stations share a common radio spectrum which is divided into channels (for instance by means of frequency division or code division techniques). Each base station has to be assigned a fixed frequency band that it uses to communicate with client systems in its vicinity. Base stations that are close to each other and transmit on overlapping frequency bands interfere which as a result yields performance drop.

The problem of assigning radio channels to base stations in telecommunication networks is well studied (examples can be found in [12, 14, 23]). However, to the best of our knowledge the existing literature only addresses the problem of assigning frequencies to a given arrangement of base stations. In this paper we show how to combine the channel assignment and access point placement problems.

The optimization objective in our facility location model is the total cost of placing the base stations. We consider various models for interference: fixed, power-dependent, and generalized. In the fixed interference model, we assume that a base-station $i$ interferes with a client $j$ if the distance between $i$ and $j$ is at most a parameter $D > 0$. In the power-dependent model, we assume that each facility can adept its transmission power and hence its coverage radius $r$. A facility now interferes with clients that are within a distance $\gamma \cdot r$ where $\gamma > 0$ is a given parameter. Finally, in the generalized interference model, we are given a function $\delta(i, j)$ that denotes the interference that a client node $j$ sees due to an open facility at node $i$. We require the total interference seen at a client node $j$ to e bounded.

We formulate the above problems as facility location problems, and obtain approximation algorithms for all of them. We also show how to combine coloring techniques from [16] with our approach in order to encompass the assignment of frequencies.

**Related Problems:** Uncapacitated facility location is MAX-SNP hard [7], and has several constant factor approximations, using local search [1, 4, 13], linear program rounding [15, 22] and primal-dual approach [10], to mention a few. In addition, there has been work on capacitated facility location, where either the capacities are hard [6, 13, 19], or where we are allowed multiple copies of a facility at a location [5, 10].

Facility location variants arise in numerous situations – purchasing cables on a network [8], constructing probabilistic Steiner trees [11], constructing multicommodity flow networks with incomplete knowledge [9], placing objects in caches with replication [2, 17], min-sum clustering [3], to name a few.

**Our Techniques:** Our algorithms are based on formulating the various problems as integer programs, and rounding the linear relaxations. Our approximations are multi-criteria, and relax the interference constraints slightly. The linear programs have unbounded integrality gap otherwise (the proofs of this fact are omitted, but follow easily from previous work [22]). In essence, we show that it is not hard to incorporate the interference and coloring constraints into any rounding scheme that preserves *locality*. Classical facility location rounding schemes [15, 22] do preserve this property. In view of the numerous applications mentioned above, we feel that understanding the power of these rounding techniques under additional constraints is of independent interest.

In the next section, we present the various models and the approximation ratios we obtain for them. As far as we are aware, we are the first to consider this variant of facility location.

## 2 The interference models

We are given a set of wireless users in a metric space with distance function $d$. We wish to place base stations to communicate with the users. The main constraint on the placement is that the base stations should not interfere. Towards this end, we wish to assign a different frequency to stations which are close to each other. We are given $\Delta$ frequencies. The goal is to place the base stations so that each user is close to a station while maintaining that any two base stations which are close to each other are transmitting on different frequencies. The optimization objective is the cost of placing the base stations.

Let's first formalize the two constraints as follows:

**Closeness Constraint:** We say a user is *close* to a base station if it is within a distance at most $r$ from it. We must assign users only to close base stations.

**Interference Constraint:** We say a base station $i$ interferes at a user $j$ if $d(i,j) \le f(i)$, where $f(i)$ is a radius that could depend on the power allocated to base station $i$. We specify $f(i)$ precisely when we consider the various formulations. The main constraint we enforce is that the total interference seen by any user $j$ on any frequency is at most $B$.

We consider several variants of this basic framework, depending on the function $f(i)$.

**Fixed Interference:** In this model, there is a cost $c_i$ associated with placing a base station at location $i$. The interference radius $f(i) = D \gg r$ for all facilities $i$, and is therefore fixed.

**Power Allocation:** Here, we consider the power allocated to the base stations in computing interference. Let the distance to the furthest user a base station $i$ is serving be $s_i$. We assume the power allocation to that station is proportional to $s_i$[1]. We then have $f(i) = \gamma \cdot s_i$ for this facility, for some $\gamma > 1$.

**General Interference:** In this model, we remove the assumption that the interference seen by a user due to a base station is either zero or one. We assume that the interference seen by user $j$ from a facility $i$ is an arbitrary number $\delta_{ij}$ if $i$ is not serving $j$, and 0 otherwise.

We present multi-criteria approximation algorithms for all these problems. The approximations are on the cost of the solution, as well as on the radii and the number of frequencies needed. For the fixed interference radius case, the approximation is constant, but for the other models, the approximations are logarithmic on the cost and number of colors needed.

## 3 Fixed interference

In this section, we optimize the cost of placing the base stations subject to the interference and closeness constraints, assuming the interference radii are fixed. We begin by formulating the integer program. Let $y_i$ denote opening a base station at location $i$. Let $f_{ij}$ denote the assignment of client $j$ to base station $i$. In the following we let $C$ denote the set of all clients.

We assume all $c_i = c$. We can remove this assumption easily by a slight modification of the rounding scheme. We also assume $B = 1$. The extension to general $B$ is mentioned at the end of this section.

[1] We could assume the power is proportional to $s_i^c$ for some constant $c$. The algorithm remains the same, but the approximation ratio gets worse.

We denote the standard linear programming relaxation of (IP1) by (LP1).

$$
\begin{aligned}
\texttt{Minimize} \quad & \sum_i c y_i && && \text{(IP1)} \\
\text{s.t} \quad & \sum_{i:d(i,j)\le r} f_{ij} \ge 1 && \forall j \in C && \texttt{(Closeness)} \\
& f_{ij} \le y_i && \forall i \in V, j \in C && (1) \\
& \sum_{i:d(i,j)\le D} y_i \le \Delta && \forall j \in C && \texttt{(Interference)} \\
& f_{ij}, y_i \in \{0,1\} && \forall i \in V, j \in C &&
\end{aligned}
$$

### 3.1 Algorithm

Algorithm 1 shows the pseudo code of our procedure. The rounding scheme is based on techniques in [22].

**Algorithm 1** The Algorithm for the Fixed Interference Model.

1: Let $(y, f)$ be a solution to (LP1)
2: **while** $\exists j \in C$ s.t. $\sum_{i:y_i<1} f_{ij} \ge 0$ **do**
3: $\quad S_j \leftarrow \{i \in V \,:\, f_{ij} > 0\}$
4: $\quad y_j \leftarrow 1$
5: $\quad y_{i'} \leftarrow 0$ for all $i' \in S_j, i' \neq j$
6: $\quad$ **for all** $j' \in C$ s.t $\exists i' \in S_j$ with $f_{i'j'} > 0$ **do**
7: $\quad\quad f_{j'j} \leftarrow 1$ and
$\quad\quad f_{j'i} \leftarrow 0$ for all $i \in C, i \neq j$
8: $\quad$ **end for**
9: **end while**
10: $\mathcal{F} \leftarrow \{i \in V \,:\, y_i > 0\}$
11: let $G = (\mathcal{F}, E)$ with
$(i_1, i_2) \in E$ iff $\exists j \in C$ s.t. $d(i_1, j) \le (D - 2r)/2$ and $d(i_2, j) \le (D - 2r)/2$
12: color $G$ with $\Delta$ colors

### 3.2 Analysis

The following lemma follow from the analysis in [22].

**Lemma 3.1.** *The following are true at the end of the algorithm:*

1. *The number of open facilities is at most the optimum number of open facilities in the LP.*
2. *All clients are served by stations that are within a radius of 2r.*

**Lemma 3.2.** *There are less then $\Delta$ facilities within $D - r$ of every client.*

*Proof.* An open base station travels no more than $r$ from $i$ to $j'$. If the radius of interference is decreased by $r$, then the base stations displaced in Step 6 will not effect the $\Delta$ bound. $\square$

**Theorem 3.3.** *Using at most $\Delta$ frequencies, all stations within $\frac{D}{2} - r$ of a client can be assigned unique frequencies.*

*Proof.* We consider two facilities adjacent if there exists a demand point within $\frac{D}{2} - r$ of both of them.

**Claim 3.4.** *Any facility is adjacent to at most $\Delta - 1$ facilities.*

*Proof.* Consider some facility $i$ that violates this claim. By our construction there has to exist a demand point $j$ such that $d(i, j) \leq r$. Now, notice that $d(i, i') \leq D - 2r$ for any neighbor $i'$ of $i$ in $G$ by triangle inequality. Hence, all of $i$'s neighbors in $G$ are within a distance of at most $D - r$ of $j$. But this means that there are more than $\Delta$ facilities within a distance of $D - r$ of $j$ which is a contradiction to Lemma 3.2. □

Since every facility is adjacent to no more than $\Delta - 1$ facilities, they can be assigned $\Delta$ frequencies using standard graph coloring methods. □

We have therefore shown an approximation algorithm that blows up closeness radii by a factor of 2, while assigning $\Delta$ frequencies in such a way that any two base stations within $\frac{D}{2} - r$ from a user have different frequencies.

We can easily add in the constraint that the number of users served by any base station is at most some number $U$. A similar rounding scheme applies to this case.

### 3.3 The Case $B > 1$

All the above results can be generalized to the case where we relax the interference constraint to say that there can be no more than $B$ stations transmitting at the same frequency within distance $D$ from any user.

In (IP1) we need to replace $\Delta$ on the right hand side of the interference constraints by $\Delta \cdot B$. We can now show that the constructed graph has maximum degree at most $\Delta \cdot B - 1$. The following theorem shows that it is possible to color the nodes of this graph using $\Delta$ colors so that any vertex is adjacent to at most $B$ vertices of the same color.

**Lemma 3.5.** *Any undirected graph $G$ with maximum degree $\Delta(G)$ can be colored with $\lceil \frac{\Delta(G)+1}{B} \rceil$ colors so that any vertex is adjacent to at most $B$ vertices of the same color.*

*Proof.* The proof can be easily deduced from [16]. We repeat it here for completeness. The coloring goes greedily – we start with an arbitrary coloring. Given a vertex which violates the coloring constraint, we flip its color to the one which is least common among its neighbors. This process decreases the number of edges both of whose end-points are of the same color, and therefore, terminates in finite time with a valid coloring. □

## 4 Power allocation

In this section we allow base stations to vary their power input and hence their coverage radii. We assume that each base station $i$ can choose an integral radius $1 \leq s_i \leq r$. The cost of opening a base station at location $i$ with radius $s$ is given by $c_i(s)$. As before, we are given an undirected graph $G = (V, E)$ and a metric on the edges of $G$.

We are also given a client set $C \subseteq V$ and a positive interference radius $D \geq r$. A feasible solution to our problem is a set of tuples

$$S = \{(i_1, s_1), \ldots, (i_k, s_k)\} \tag{2}$$

such that for each $j \in C$ there is an $(i,r) \in S$ with $d(j,i) \leq r$. Moreover, we want to assign frequencies $\delta_i \in \{1,\ldots,\Delta\}$ to open facilities from $S$ such that there are at most $B > 0$ open base stations close to any client $j$ that transmit on the same frequency. More formally, a base station $i$ that is open with radius $s$ *interferes with* $j$ if $d(i,j) \leq \gamma \cdot s$. Here, $\gamma > 1$ is an interference parameter. Let $\mathcal{F}$ be the set open facility, radius pairs for a feasible solution. We then require the set

$$\{(i,s) \in \mathcal{F} \,:\, d(i,j) \leq \gamma \cdot s,\, \delta_i = l\} \tag{3}$$

to have cardinality at most $B$ for all $j \in C$ and for all $1 \leq l \leq \Delta$. Our ultimate goal is to find a feasible set $S$ and an assignment $\delta$ of frequencies such that the total facility cost is minimized.

### 4.1 An IP formulation

We can model the above problem as an integer linear program. The IP has a variable $y_{i,s}$ for each facility location $i$ and each radius $1 \leq s \leq r$. This variable has value 1 iff facility $i$ is opened at radius $s$.

We can now state the integer program for this model. We let (LP2) denote the standard linear programming relaxation of (IP2).

$$\begin{aligned}
\texttt{Minimize} \quad & \sum_{i\in V}\sum_{s=1}^{r} c_i(s) y_{i,s} & & \text{(IP2)}\\
\text{s.t} \quad & \sum_{s=1}^{r}\sum_{i:d(i,j)\leq s} y_{i,s} \geq 1 & \forall j \in C \quad & (4)\\
& \sum_{s=1}^{r}\sum_{i:d(i,j)\leq \gamma\cdot s} y_{i,s} \leq \Delta \cdot B & \forall j \in C \quad & (5)\\
& y_{i,s} \in \{0,1\} & \forall i, s &
\end{aligned}$$

### 4.2 A polynomial-time solvable special case

In this section we assume $\Delta = B = 1$. That is, we do not allow any overlap and furthermore restrict ourselves to only one frequency. Furthermore, we will assume that $\gamma \geq 3$. It turns out that (LP2) has only integral extreme points in this case.

**Theorem 4.1.** *(LP2) has only integral extreme points whenever $\Delta = B = 1$ and $\gamma \geq 3$.*

*Proof.* We assume for the sake of contradiction that there is a fractional solution $y^0$ to (LP2). In the following, we construct two feasible solutions $y^1$ and $y^2$ such that

$$y^0 = 1/2 \cdot y^1 + 1/2 \cdot y^2$$

contradicting the fact that $y^0$ is an extreme point of the rational polyhedron described by (LP2).

Let $i_0$ be a fractionally open facility of largest radius, i.e. there is an $s_0 > 0$ such that $0 < y^0_{i_0,s_0} < 1$ and there is no $s' > s_0$ and $i'$ such that $0 < y^0_{i',s'} < 1$. Also we let $B(i_0,s)$ denote the set of vertices $i$ such that $d(i_0,i) \leq s$. Furthermore, we define $B(i_0,s_1,s_2)$ to be the set of vertices $i$ such that $s_1 < d(i_0,i) \leq s_2$.

For a node $j \in C \cap B(i_0, s_0)$ we define the set $\mathcal{F}_j$ of facility, radius pairs that fractionally serve client $j$, i.e.

$$\mathcal{F}_j = \{(i,s) \,:\, y^0_{i,s} > 0,\, d(i,j) \leq s\}.$$

We now choose $\epsilon \leq y^0_{i_0,s_0}$ and define $y^1$ and $y^2$ as follows

$$y^1_{i,s} = \begin{cases} y^0_{i,s} - \epsilon & : \quad i = i_0 \text{ and } s = s_0 \\ \left(1 + \frac{\epsilon}{1 - y^0_{i_0,s_0}}\right) y^0_{i,s} & : \quad \exists j \in B(i_0, s_0) \text{ such that } (i,s) \in \mathcal{F}_j \\ y^0_{i,s} & : \quad \text{otherwise} \end{cases}$$

$$y^2_{i,s} = \begin{cases} y^0_{i,s} + \epsilon & : \quad i = i_0 \text{ and } s = s_0 \\ \left(1 - \frac{\epsilon}{1 - y^0_{i_0,s_0}}\right) y^0_{i,s} & : \quad \exists j \in B(i_0, s_0) \text{ such that } (i,s) \in \mathcal{F}_j \\ y^0_{i,s} & : \quad \text{otherwise} \end{cases}$$

Notice that $y^0 = 1/2 \cdot y^1 + 1/2 \cdot y^2$ and hence, we are done once we have shown the feasibility of the the above two solutions.

In the following we establish two claims which turn out to be useful in the feasibility proof.

**Claim 4.2.** *Let $y^0$ be a feasible solution to (LP2) such that $(i_0, s_0)$ is the fractionally opened facility of largest radius. Let $\mathcal{F}$ be the set of pairs $(i,s)$ such that $y^0_{i,s} > 0$. Then there does not exist a client node $j \in C \setminus B(i_0, 2s_0)$ and an $(i,s) \in \mathcal{F}$ such that $i \in B(i_0, s_0)$ and $d(i,j) \leq \gamma \cdot s$.*

*Proof.* Assume for the sake of contradiction that there is a $j \in C \setminus B(i_0, 2s_0)$ such that $d(i,j) \leq \gamma \cdot s$ for some $(i,s) \in \mathcal{F}$ and $i \in B(i_0, s_0)$. Clearly, since $j \notin B(i_0, 2s_0)$, $(i,s)$ cannot serve $j$. On the other hand, $(i,s)$ interferes with $j$ and so does every facility that serves $j$. Hence, it follows from (4) that node $j$ sees more than one unit of interference. A contradiction. □

The following claim has a very similar flavor.

**Claim 4.3.** *Let $y^0$ be a feasible solution to (LP2) such that $(i_0, s_0)$ is the fractionally opened facility of largest radius. Let $j \in C \cap B(i_0, s_0)$ be a client node. We must have*

$$\sum_{(i,s) \in \mathcal{F}_j} y^0_{i,s} = 1.$$

*Furthermore, there cannot exist a fractionally open facility $(i,s) \notin \mathcal{F}_j$ such that $d(i,j) \leq \gamma \cdot s$.*

*Proof.* Clearly, by feasibility of $y^0$ there must be at least one facility in $\mathcal{F}_j$. On the other hand, each facility serving $j$ interferes with $j$. This proves the first part of the claim.

The second part of the claim follows from the fact that the existence of an $(i,s) \notin \mathcal{F}_j$ with $d(i,j) \leq \gamma \cdot s$ together with (4) would imply that node $j$ sees more than one unit of interference. □

We continue with the proof of feasibility of $y^1$ and $y^2$. It is straight forward to verify that both $y^1$ and $y^2$ satisfy the demand constraints (4).

We show that $y^1$ satisfies all constraints of type (5). It follows from Claim 4.2 that none of the interference constraints for nodes $j \in C \setminus B(i_0, 2s_0)$ are violated by $y^1$. Also, by our definition of $y^1$ we do not increase the value of any $y^0_{i,s}$ such that $i \in B(i_0, s_0, 2s_0)$. Hence, it suffices to show that $y^1$ does not violate any of the constraints (5) corresponding to $j \in B(i_0, s_0)$.

Let $j \in B(i_0, s_0) \cap C$ be an arbitrary client node. It follows from Claim 4.3 that $y^1$ inflicts a total interference of

$$\sum_{(i,s)\in\mathcal{F}_j} y^1_{i,s}$$

at node $j$. We can rewrite the last expression in terms of $y^0$ as

$$y^0_{i_0,s_0} - \epsilon + \frac{\epsilon}{1 - y^0_{i_0,s_0}} \cdot \sum_{\substack{(i,s)\in\mathcal{F}_j \\ (i,s)\neq(i_0,s_0)}} y^0_{i,s} + \sum_{\substack{(i,s)\in\mathcal{F}_j \\ (i,s)\neq(i_0,s_0)}} y^0_{i,s}$$

It now follows from Claim 4.3 that the second and third term in the last sum cancel out and hence the total interference seen at node $j$ does not increase in solution $y^1$.

Proving that $y^2$ satisfies all interference constraints is completely analogous to the above proof.

□

### 4.3 Arbitrary $\Delta$ and $B$

The algorithm for the case of arbitrary integers $\Delta$ and $B$ is based on randomized rounding [21]. Algorithm 2 shows the pseudo code of the algorithm for this case. We first compute a set $\mathcal{F}$ of facilities and corresponding radii. Later, we assign frequencies to the access points in $\mathcal{F}$. The details for the coloring step are hidden behind a call to function `color`. This function assigns frequencies to facilities such that the conditions stated in (3) are satisfied.

**Algorithm 2** The algorithm for the power allocation model with bounded overlap: `PowerAlloc`

Let $y$ be a solution to (LP2)
**repeat**
  **for all** $i \in V, 1 \leq s \leq r$ **do**
    **if** $y_{i,s} \geq 1/(c \log k)$ **then**
      $y_{i,s} \leftarrow 1$
    **else**
      $y_{i,s} \leftarrow 1$ with probability $y_{i,s} \cdot c \log k$.
    **end if**
  **end for**
  $\mathcal{F} \leftarrow \{(i,s) : y_{i,s} = 1\}$
  $I_j \leftarrow \{(i,s) \in \mathcal{F} : d(i,j) \leq \gamma \cdot s\}$ for all $j \in C$
  `apx` $\leftarrow \sum_{(i,s)\in\mathcal{F}} c_i(s)$
**until** `apx` $\leq$ `opt`$_{LP} \cdot c \log k$ **and** $|I_j| \leq c \log k \cdot \Delta B$ for all $j \in C$
$\delta \leftarrow$ `color`$(\mathcal{F}, B)$
`return`$(\mathcal{F}, \delta)$

#### 4.3.1 Frequency assignment

The frequency assignment step in Algorithm 2 needs to differ from earlier approaches used in Section 3. To see this, let $\mathcal{F}$ be the set of facilities computed by the above algorithm and assume $B = 1$ for now.

We let $G$ be a graph with node set $\mathcal{F}$ and connect two facilities $(i_1, s_1)$ and $(i_2, s_2)$ by an edge whenever there exists a client node $j$ such that

$$d(i_1, j) \leq \beta \cdot s_1 \text{ and } d(i_2, j) \leq \beta \cdot s_2$$

where $\beta$ is a parameter that depends on $\gamma$. The main idea in Theorem 3.3 that enabled us to color the nodes of $G$ using at most $\Delta$ colors was to bound the maximum node degree by $\Delta - 1$.

However, this turns out to be impossible here, for any $\beta$. In particular, a client node $j \in C$ served by $(i_1, s_1)$ might not see any interference at all from $(i_2, s_2)$. On the other hand, it is conceivable that there are nodes that see interference from both, $(i_1, s_1)$ and $(i_2, s_2)$ (e.g. consider the case where $s_1 \gg s_2$ and $j$ is close to $i_1$).

Now, let $\beta = (\gamma - 1)/2$ in the definition of the auxiliary graph $G$ above. It turns out that directing the edges of $G$ helps. We obtain the digraph $G^d$ from $G$ by replacing an edge between nodes $(i_1, s_1)$ and $(i_2, s_2)$ by an arc from $(i_1, s_1)$ to $(i_2, s_2)$ if $s_1 > s_2$ and by an arc from $(i_2, s_2)$ to $(i_1, s_1)$ whenever $s_2 > s_1$. Whenever $s_1$ equals $s_2$ we replace the edge by a pair of anti-parallel arcs.

Now, define a valid *B-coloring* $G^d$ to be a function $\delta_v$ on the node set such that for each node $(i_0, s_0) \in V[G^d]$ the set

$$\{(i, s) \in \Gamma^{-1}(i_0, s_0) \,:\, \delta_{(i,s)} = \delta_{(i_0,s_0)}\}$$

has cardinality at most $B - 1$. Here, $\Gamma^{-1}(v)$ is the set of nodes $u \in V[G^d]$ such that $(u, v)$ is an arc in $G^d$. In other words, each node $v \in V[G^d]$ has at most $B - 1$ predecessors of color $\delta_v$.

Let $\Delta^-(G^d) = \max_{v \in \mathcal{F}} |\Gamma^{-1}(v)|$. The proof of the following lemma crucially uses the direction of arcs in $G^d$ to infer a coloring ordering and is otherwise similar to the proof of Lemma 3.5. We omit it.

**Lemma 4.4.** *There is a a feasible* $\lceil \Delta^{-1}(G^d)/B \rceil + 1$ *coloring of* $G^d$.

The following lemma bounds the in-degree of any node $i \in V$ in $G^d$.

**Lemma 4.5.** *For any* $i \in V$ *that* $\Gamma^{-1}(i)$ *has cardinality at most* $c \log k \cdot \Delta \cdot B - 1$.

*Proof.* Assume for the sake of contradiction that there is a facility $i_0 \in V$ such that

$$|\Gamma^{-1}(i_0)| > c \log k \cdot \Delta B. \tag{6}$$

Let $j \in C$ be a node served by $i_0$ and let $I_j$ be defined as in step 11 of Algorithm 2.

Let $(i, s) \in \Gamma^{-1}(i_0, s_0)$ be an arbitrary predecessor of $(i_0, s_0)$. Since $G^d$ contains an arc from $(i, s)$ to $(i_0, s_0)$ there must exist a client node $j_0$ such that $d(i, j_0) \leq \frac{\gamma-1}{2} s$ and $d(i_0, j_0) \leq \frac{\gamma-1}{2} s_0$.

From triangle inequality and from the fact that $j$ is within $s_0$'s coverage radius we obtain that

$$d(i, j) \leq d(i, j_0) + d(i_0, j_0) + d(i_0, j) \leq \frac{\gamma - 1}{2} s + \frac{\gamma - 1}{2} s_0 + s_0$$

Since $G^d$ has an arc from $(i, s)$ to $(i_0, s_0)$ we also know that $s \geq s_0$. Hence, it follows from the above inequality that

$$d(i, j) \leq \gamma \cdot s.$$

In other words, $j$ sees interference from $i$. In fact, $j$ sees interference from all predecessors of $(i_0, s_0)$ and from $(i_0, s_0)$ itself. But this is a contradiction to the termination condition $|I_j| \leq c \log k \cdot \Delta B$ imposed in step 13 of Algorithm 2. □

It is now immediate that there exists a feasible $B$-coloring using at most $c \log k \cdot \Delta$ colors. A feasible $B$-coloring $\{\delta_i\}_{i \in V}$ of $G^d$ gives rise to a feasible frequency assignment.

**Lemma 4.6.** *Let $\mathcal{F}$ be the set of facility radius pairs produced by Algorithm 2 and let the auxiliary digraph $G^d$ be defined as before. Furthermore, let $\{\delta_i\}_{i\in V}$ be the feasible B-coloring of $G^d$. Then, for any client node $j \in C$ and for any $1 \le l \le c \log k \cdot \Delta$, the set*

$$I_j^l := \{(i,s) \in \mathcal{F} \,:\, d(i,j) \le \frac{\gamma - 1}{2} \cdot s,\ \delta_i = l\}$$

*must have cardinality at most $B$.*

*Proof.* Let $j \in C$ be an arbitrary client node and let $(i_0, s_0)$ be the facility of smallest radius such that $d(i_0, j) \le \frac{\gamma-1}{2} s_0$. Clearly, for all other $(i,s) \in I_j^l$ there must be an arc in $G^d$ pointing from $(i,s)$ to $(i_0, s_0)$. It now follows that $I_j^l$ can have cardinality at most $B$. □

### 4.3.2 Analysis

In order to show correctness of Algorithm 2 it suffices to prove that steps 3 to 12 are not repeated too often. The proofs of the following lemmas are relegated to Appendix A.

**Lemma 4.7.** *Let $\mathcal{F}$ be the set of facility, radius pairs produced by the rounding procedure outlined in the above algorithm. Then, with probability at least $1 - k^{1-c}$ for each client node $j$, there is a $(i,s) \in \mathcal{F}$ such that $d(i,j) \le s$.*

**Lemma 4.8.** *Let $\mathcal{F}$ be the set of facility, radius pairs produced by our rounding procedure. Then, with probability at least $1 - k^{-c/2}$, the sets $I_j$ have cardinality at most $c \log k \cdot \Delta B$ for each client node $j \in C$.*

It is now easy to see that we can choose $c > 2$ such that conditions (1) and (2) from above are met. Lemma 4.7 and 4.8 now yield the following theorem:

**Theorem 4.9.** *In the power allocation setting there exists an algorithm that computes a set of facility, radius pairs $\mathcal{F}$ and a frequency assignment $\{\delta_i\}_{i\in V}$ such that*

1. $\sum_{(i,s)\in\mathcal{F}} c_i(s) \le c \log k \cdot \texttt{opt}$ *where* $\texttt{opt}$ *denotes the cost of an optimum solution*
2. *each client node $j \in C$ has a facility $(i,s) \in \mathcal{F}$ such that $d(i,j) \le s$*
3. *for each client node $j \in C$ and for each $1 \le l \le c \log k \cdot \Delta$, the set*

$$\{(i,s) \in \mathcal{F} \,:\, d(i,j) \le \frac{\gamma - 1}{2} s,\ \delta_i = l\}$$

*has cardinality at most $B$.*

*Furthermore, in the special case of $\Delta = B = 1$, we can solve the problem exactly in polynomial time.*

## 5 Generalized interference

In this section we want to place facilities so that for each client node $j \in C$ there exists an open facility $i_j$ within distance at most $r$. A client node $j$ sees interference from each open facility $i$ different from $i_j$. The amount of interference that $j$ sees from $i$ is given by a function $\delta_{i,j}$. So, for example, the interference that node $j$ sees from $i$ might depend on the distance between $i$ and $j$ in the underlying graph $G$. We want the total interference felt at node $j$ to be at most $B > 0$. We also incur a cost of $c$ for placing a facility. The goal is to minimize placement cost while obeying the interference constraints at each node $j \in C$.

We assume $\Delta = 1$ in the discussion below. The generalization to arbitrary $\Delta$ easily follows from the discussion in the previous sections.

Our algorithm consists of two phases. The first phase partitions the vertex set $V$ into subsets $V_1, \ldots, V_k$. We prove that the optimum solution needs to place at least one facility in each of these sets. As before, we let $B(v, r)$ denote the ball of radius $r$ centered at $v$.

1. $l \leftarrow 0$

2. Repeat as long as $C \neq \emptyset$: let $j \in C$ be an arbitrary client node. Let $V_l \leftarrow B(j, 2r)$ and set $l \leftarrow l + 1$. Also, let $C \leftarrow C \setminus B(j, 2r)$.

In the second phase we place exactly one facility $i_l$ in each $V_l$ for $1 \leq l \leq k$. Notice that a client node $j \in V_l$ sees interference from each of the facilities $i_{l'}$ for $l' \neq l$. Our goal is to place $i_1, \ldots, i_k$ such that the interference at each node $j \in C$ is at most $B$.

For each $V_l$ and for each $i \in V_l$ we define an *interference vector* $\delta_{l,i}$ whose $j^{th}$ component is the interference that node $j$ sees if $i$ is open.

$$\delta_{l,i}(j) = \begin{cases} 0 & : \ j \in V_l \\ \delta_{i,j} & : \ j \notin V_l \end{cases}$$

We can write the placement problem as an integer program. As before, we have binary variables $y_i$ where $y_i = 1$ iff we place a facility at node $i$.

$$\begin{aligned} \texttt{Minimize} \quad & \sum_{i \in V} c \cdot y_i & & \text{(IP3)} \\ \text{s.t} \quad & \sum_{1 \leq l \leq k} \sum_{i \in V_l} \delta_{l,i}(j) \cdot y_i \leq B & \forall j \in C & \\ & \sum_{i \in V_l} y_i = 1 & \forall 1 \leq l \leq k & \\ & y_i \in \{0, 1\} & \forall i & \end{aligned}$$

As before, we will use (LP3) to denote the linear programming relaxation of (IP3). In the following, we again let $k$ denote the cardinality of the client set. We state the second phase of our algorithm:

3. Solve (LP3) $\rightarrow y$.

4. For all $i \in V$ such that $y_i \leq 1/(c \log k)$ where $c$ is a positive constant to be specified later. Remove $B(i, 2r)$ from $V$ and round $y_i$ to one. Let $V'$ be the set of remaining vertices.

5. For all $i \in V'$, let $\overline{y}_i \leftarrow c \log k \cdot y_i$.

6. Open facility $i$ with probability $\overline{y}_i$. Let $\mathcal{F}$ be the set of open facilities.

7. Repeat the rounding process (i.e. go back to step (6)) if either

   (a) there is an $1 \leq l \leq k$ such that $V_l \cap \mathcal{F} = \emptyset$ or

   (b) $\sum_{1 \leq l \leq k} \sum_{i \in V_l} \delta_{l,i}(j) \cdot y_i > c \log k \cdot B$ for any $j \in C$

8. For each $1 \leq l \leq k$, close all but one facility in $V_l$.

The following theorem is proved in Appendix B:

**Theorem 5.1.** *In the generalized interference setting, there is an algorithm that opens a set $\mathcal{F}$ of facilities such that*

1. $c|F| \leq \texttt{opt}$ *where* $\texttt{opt}$ *denotes the cost of an optimum solution*
2. *each client* $j \in C$ *is at distance at most* $2r$ *from an open facility*
3. *each client sees interference at most* $c \log k \cdot B$.

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# A Proofs for Lemmas 4.7 and 4.8

**Lemma A.1.** *Let $\mathcal{F}$ be the set of facility, radius pairs produced by the rounding procedure outlined in the above algorithm. Then, with probability at least $1 - k^{1-c}$ for each client node $j$, there is a $(i, s) \in \mathcal{F}$ such that $d(i, j) \leq s$.*

*Proof.* Let $y$ be a solution to (LP2) and let $\overline{y}$ denote the scaled solution $c \cdot \log k \cdot y$. Let $j \in C$ be a client node and let $\mathcal{F}_j$ be the set of facilities that fractionally serve $j$. Let $E_j$ denote the event that none of the facilities in $\mathcal{F}_j$ are open. The probability of $E_j$ can be bounded by

$$P_j = \Pi_{(i,s)\in\mathcal{F}_j}(1 - \overline{y}_{i,s}). \tag{7}$$

Since $y$ is a feasible solution to (LP2) we know that

$$\sum_s \sum_{i:d(i,j)\leq s} \overline{y}_{i,s} \geq c \cdot \log k.$$

Let $k_j$ be the number of facility, radius pairs that fractionally serve client node $j$. It is not hard to see that (7) is maximized if $y_{i,s} = c \cdot \log k / k_j$. In this case, we see that

$$P_j = \left(1 - \frac{c \log k}{k_j}\right)^{k_j} \leq \left(\frac{1}{k}\right)^c.$$

An application of the union bound implies that the probability of the event $E = \bigcup_{j\in C} E_j$ can be bounded by $k^{1-c}$. □

The following lemma bounds the probability that the interference at any node is high.

**Lemma A.2.** *Let $\mathcal{F}$ be the set of facility, radius pairs produced by our rounding procedure. Then, with probability at least $1 - k^{-c/2}$, the sets $I_j$ have cardinality at most $c \log k \cdot \Delta B$ for each client node $j \in C$.*

*Proof.* Again, let $y$ be the solution to (LP2) and let $\overline{y}$ denote the scaled solution $c \cdot \log k \cdot y$. For a client node $j \in C$, let $E_j$ be the event that the set $I_j$ has cardinality more than $c \cdot \log k \cdot \Delta B$. Let $E$ be the union of the $E_j$ events.

We introduce a Bernoulli variable $X^j_{i,s}$ that has value 1 with probability $\overline{y}_{i,s}$ and 0 otherwise. The cardinality of the set $I_j$ can now be expressed by another variable

$$X^j = \sum_{\substack{(i,s)\in\mathcal{F} \\ d(i,j)\leq\gamma\cdot s}} X^j_{i,s}.$$

Let $\mu_j$ be the expected value of $X^j$. Since $y$ is feasible for (LP2) it can be readily seen that $\mu_j \leq c \cdot \log k \cdot \Delta B$. Hence, an application of the Chernoff-Hoeffding bound (for a proof see e.g. [18]) yields

$$\texttt{Pr}[E_j] = \texttt{Pr}[X^j - \mu_j > c \cdot \log k \cdot \Delta B] \leq e^{-\mu_j/2} = \frac{1}{k^{c/2}}.$$

Another application of the union bound yields the wanted result. □

It is now easy to see that we can choose $c > 2$ such that conditions (1) and (2)

## B Analysis of generalized interference algorithm

Let $k$ be defined as in Section 5. Then, it follows from a k-center like argument (see for instance [20]) that each feasible solution needs to open at least $k$ facilities. The following lemma is immediate.

**Lemma B.1.** *Let $\mathcal{F}$ be the set of facilities computed by the above algorithm and let* `opt` *be the facility cost incurred by an optimum solution. We then must have $c \cdot |\mathcal{F}| \leq$* `opt`*.*

Since the final solution opens a facility in each of the $V_l$, we also have an open facility close to each client node $j \in C$.

**Lemma B.2.** *Let $\mathcal{F}$ be the set of facilities opened by our algorithm. Then, for each $j \in C$ there must exist an $i \in \mathcal{F}$ such that $d(i, j) \leq 2r$.*

All that remains is showing that phase 2 of the above algorithms terminates in polynomial time. As in the last section it suffices to prove that with at least constant probability, neither case (a) nor (b) in step 7 of the above algorithm occurs.

First, using similar arguments as in Lemma 4.7 we can show the following result.

**Lemma B.3.** *Let $\mathcal{F}$ be the set of facilities opened by the rounding procedure in the above algorithm. Then, with probability at least $1 - n^{1-c}$ for each client node $1 \leq l \leq k$, $V_l \cap \mathcal{F}$ is non-empty.*

Similarly, we can show that we can choose $c$ so that there is at least a constant probability that none of the the client nodes sees more than $c \log k \cdot B$ interference.

**Lemma B.4.** *Let $\mathcal{F}$ be the set of facilities opened by our rounding procedure. Then, with probability at least $1 - n^{-c/2}$, for each client node $j \in C$, we have*

$$\sum_{i \in \mathcal{F}} \delta_{i,j} \leq c \log k \cdot B.$$

The techniques used in the proof of the last theorem resemble those used in Lemma 4.8. The following theorem summarizes the analysis.

**Theorem B.5.** *In the generalized interference setting, there is an algorithm that opens a set $\mathcal{F}$ of facilities such that*

1. *$c|F| \leq$ `opt` where `opt` denotes the cost of an optimum solution*
2. *each client $j \in C$ is at distance at most $2r$ from an open facility*
3. *each client sees interference at most $c \log k \cdot B$.*