# CS156: The Calculus of Computation

Zohar Manna

Winter 2008

Lecturer:
Zohar Manna (manna@cs.stanford.edu)

TAs:
Ben Newman (benjamin@cs.stanford.edu)
Office Hours: MW 12:30-2:30 in Gates B24A
Eric Smith (ewsmith@stanford.edu)
Office Hours: MTh 2:30-4:30 in Gates 312

# Calculus of Computation?

> *It is reasonable to hope that the relationship between* **computation** *and* **mathematical logic** *will be as fruitful in the next century as that between* **analysis** *and* **physics** *in the last. The development of this relationship demands a concern for both applications and mathematical elegance.*

John McCarthy
*A Basis for a Mathematical Theory of Computation*, 1963

# Grading

- Homeworks (60%)
  - weekly
  - no late assignments
  - no collaboration
- Final Exam (40%)
  - open book and notes

# Contact

- Course administration: send email to Ben
- Questions about course content: send email to Ben, Eric
- Text corrections: send email to Prof. Manna, Ben, and Eric

# Assignment #1 (due Tue, Jan 15)

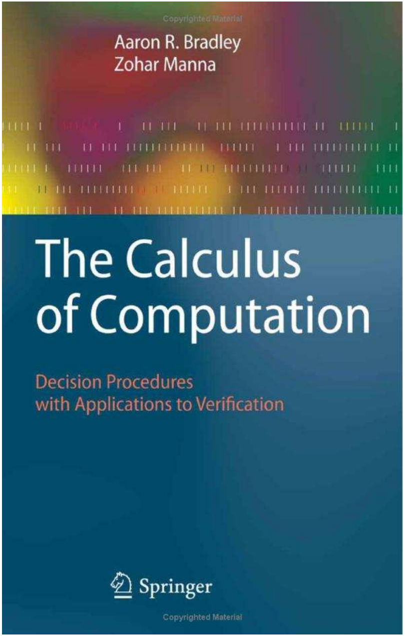
- 1.1 b, c
- 1.2 w, t
- 1.3 (note typo: the last $\vee$ should be a $\wedge$)
- 1.5 c
- 1.8 a, b

# Textbook

THE CALCULUS OF COMPUTATION:
Decision Procedures with
Applications to Verification

by
Aaron Bradley
Zohar Manna

Springer 2007


Copyrighted Material
Aaron R. Bradley
Zohar Manna
The Calculus of Computation
Decision Procedures
with Applications to Verification
Springer
Copyrighted Material

# Topics: Overview

1. First-Order logic
2. Specification and verification
3. Satisfiability decision procedures
4. Static analysis

# Part I: Foundations

1. Propositional Logic
2. First-Order Logic
3. First-Order Theories
4. Induction
5. Program Correctness: Mechanics
   Inductive assertion method, Ranking function method
6. Program Correctness: Strategies

# Part II: Algorithmic Reasoning

7. Quantified Linear Arithmetic
   Quantifier elimination for integers and rationals
8. Quantifier-Free Linear Arithmetic
   Linear programming for rationals
9. Quantifier-Free Equality and Data Structures
10. Combining Decision Procedures
    Nelson-Oppen combination method
11. Arrays
    More than quantifier-free fragment
12. Invariant Generation

# Motivation I

Decision Procedures are algorithms to decide formulae.
These formulae can arise

- in software verification.
- in hardware verification

Consider the following program:

---

for

@ $\ell \leq i \leq u \wedge (rv \;\leftrightarrow\; \exists j.\, \ell \leq j < i \wedge a[j] = e)$

$(\texttt{int}\ i := \ell;\ i \leq u;\ i := i + 1)\ \{$

$\texttt{if}\ (a[i] = e)\ rv := \texttt{true};$

$\}$

---

How can we decide whether the formula is a loop invariant?

# Motivation II

Prove:

---

$\texttt{assume}\ \ell \leq i \leq u \land (rv \leftrightarrow \exists j.\ \ell \leq j < i \land a[j] = e)$

$\texttt{assume}\ i \leq u$

$\texttt{assume}\ a[i] = e$

$rv := \texttt{true};$

$i := i + 1$

$\texttt{@}\ \ell \leq i \leq u \land (rv \leftrightarrow \exists j.\ \ell \leq j < i \land a[j] = e)$

---

# Motivation III

$$
\begin{array}{l}
\texttt{assume}\ \ell \le i \le u \land (rv \;\leftrightarrow\; \exists j.\ \ell \le j < i \land a[j] = e) \\
\texttt{assume}\ i \le u \\
\texttt{assume}\ a[i] \neq e \\
i := i + 1 \\
\texttt{@}\ \ell \le i \le u \land (rv \;\leftrightarrow\; \exists j.\ \ell \le j < i \land a[j] = e)
\end{array}
$$

A Hoare triple {P} S {Q} holds, iff

$$P \;\rightarrow\; wp(S, Q)$$

(wp denotes "weakest precondition")

# Motivation IV

For assignments wp is computed by substitution:

$$\begin{array}{l}
\texttt{assume } \ell \leq i \leq u \land (rv \leftrightarrow \exists j.\, \ell \leq j < i \land a[j] = e) \\
\texttt{assume } i \leq u \\
\texttt{assume } a[i] = e \\
rv := \texttt{true}; \\
i := i + 1 \\
\texttt{@ } \ell \leq i \leq u \land (rv \leftrightarrow \exists j.\, \ell \leq j < i \land a[j] = e)
\end{array}$$

Substituting $\top$ for rv and $i+1$ for $i$, the postcondition (denoted by the @ symbol) holds if and only if:

$$\begin{array}{rl}
& \ell \leq i \leq u \land (rv \leftrightarrow \exists j.\, \ell \leq j < i \land a[j] = e) \land i \leq u \land a[i] = e \\
\rightarrow & \ell \leq i+1 \leq u \land (\top \leftrightarrow \exists j.\, \ell \leq j < i+1 \land a[j] = e)
\end{array}$$

# Motivation V

We need an algorithm that decides whether this formula holds. If the formula does not hold, the algorithm should give a counterexample; e.g.,

$$\ell = 0, i = 1, u = 1, rv = \mathit{false}, a[0] = 0, a[1] = 1, e = 1.$$

We will discuss such algorithms in later lectures.

# CS156: The Calculus of Computation

Zohar Manna

Winter 2008

## Chapter 1: Propositional Logic (PL)

# Propositional Logic (PL)

## PL Syntax

| | |
|---|---|
| Atom | truth symbols $\top$ ("true") and $\bot$ ("false") |
| | propositional variables $P, Q, R, P_1, Q_1, R_1, \ldots$ |
| Literal | atom $\alpha$ or its negation $\neg\alpha$ |
| Formula | literal or application of a |
| | logical connective to formulae $F, F_1, F_2$ |

| | | |
|---|---|---|
| $\neg F$ | "not" | (negation) |
| $F_1 \wedge F_2$ | "and" | (conjunction) |
| $F_1 \vee F_2$ | "or" | (disjunction) |
| $F_1 \rightarrow F_2$ | "implies" | (implication) |
| $F_1 \leftrightarrow F_2$ | "if and only if" | (iff) |

Example:

formula $F : (P \wedge Q) \;\rightarrow\; (\top \vee \neg Q)$
atoms: $P,\ Q,\ \top$
literals: $P,\ Q,\ \top,\ \neg Q$
subformulae: $P,\ Q,\ \top,\ \neg Q,\ P \wedge Q,\ \top \vee \neg Q,\ F$
abbreviation

$$F : P \wedge Q \;\rightarrow\; \top \vee \neg Q$$

# PL Semantics (meaning of PL)

Formula $F$ + Interpretation $I$ = Truth value (true, false)

Interpretation

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}, \cdots\}$$

Evaluation of $F$ under $I$:

| $F$ | $\neg F$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

where 0 corresponds to value false
1 true

| $F_1$ | $F_2$ | $F_1 \wedge F_2$ | $F_1 \vee F_2$ | $F_1 \rightarrow F_2$ | $F_1 \leftrightarrow F_2$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |

Example:

$F : P \wedge Q \;\rightarrow\; P \vee \neg Q$
$I : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ i.e., $I[P] = \text{true}, I[Q] = \text{false}$

| $P$ | $Q$ | $\neg Q$ | $P \wedge Q$ | $P \vee \neg Q$ | $F$ |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 1 |

$1 = \text{true}$ $0 = \text{false}$

$F$ evaluates to true under $I$; i.e., $I[F] = \text{true}$.

# Inductive Definition of PL's Semantics

$$\begin{array}{lllll} I & \models & F & \text{if } F \text{ evaluates to} & \text{true} \quad \text{under } I \\ I & \not\models & F & & \text{false} \end{array}$$

<u>Base Case</u>:

$$I \models \top \qquad I \not\models \bot$$

$$\begin{array}{lllll} I & \models & P & \text{iff} & I[P] = \text{true; i.e., } P \text{ is true under } I \\ I & \not\models & P & \text{iff} & I[P] = \text{false} \end{array}$$

<u>Inductive Case</u>:

$$\begin{array}{llll} I & \models & \neg F & \text{iff } I \not\models F \\ I & \models & F_1 \wedge F_2 & \text{iff } I \models F_1 \text{ and } I \models F_2 \\ I & \models & F_1 \vee F_2 & \text{iff } I \models F_1 \text{ or } I \models F_2 \text{ (or both)} \\ I & \models & F_1 \rightarrow F_2 & \text{iff } I \models F_1 \text{ implies } I \models F_2 \\ I & \models & F_1 \leftrightarrow F_2 & \text{iff, } I \models F_1 \text{ and } I \models F_2, \\ & & & \quad \text{or } I \not\models F_1 \text{ and } I \not\models F_2 \end{array}$$

<u>Note</u>:

$I \models F_1 \rightarrow F_2$ iff $I \not\models F_1$ or $I \models F_2$.

$I \not\models F_1 \rightarrow F_2$ iff $I \models F_1$ and $I \not\models F_2$.

$I \not\models F_1 \vee F_2$ iff $I \not\models F_1$ and $I \not\models F_2$.

Example of Inductive Reasoning:

$$F: \; P \wedge Q \;\rightarrow\; P \vee \neg Q$$

$$I: \; \{P \;\mapsto\; \text{true}, \;\; Q \;\mapsto\; \text{false}\}$$

| | | | | | |
|---|---|---|---|---|---|
| 1. | $I$ | $\models$ | $P$ | since $I[P] = \text{true}$ | |
| 2. | $I$ | $\not\models$ | $Q$ | since $I[Q] = \text{false}$ | |
| 3. | $I$ | $\models$ | $\neg Q$ | by 2 and $\neg$ | |
| 4. | $I$ | $\not\models$ | $P \wedge Q$ | by 2 and $\wedge$ | |
| 5. | $I$ | $\models$ | $P \vee \neg Q$ | by 1 and $\vee$ | |
| 6. | $I$ | $\models$ | $F$ | by 4 and $\rightarrow$ | Why? |

Thus, $F$ is true under $I$.

Note: steps 1, 3, and 5 are nonessential.

# Satisfiability and Validity

$F$ satisfiable iff there exists an interpretation $I$ such that $I \models F$.
$F$ valid iff for all interpretations $I$, $I \models F$.

$$\boxed{F \text{ is valid iff } \neg F \text{ is unsatisfiable}}$$

Goal: devise an algorithm to decide validity or unsatisfiability of formula $F$.

# Method 1: Truth Tables

Example $F : P \wedge Q \;\rightarrow\; P \vee \neg Q$

| $P$ | $Q$ | $P \wedge Q$ | $\neg Q$ | $P \vee \neg Q$ | $F$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 |

Thus $F$ is valid.

Example $F : P \vee Q \;\rightarrow\; P \wedge Q$

| $P$ | $Q$ | $P \vee Q$ | $P \wedge Q$ | $F$ | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | $\leftarrow$ satisfying $I$ |
| 0 | 1 | 1 | 0 | 0 | $\leftarrow$ falsifying $I$ |
| 1 | 0 | 1 | 0 | 0 | |
| 1 | 1 | 1 | 1 | 1 | |

Thus $F$ is satisfiable, but invalid.

# Method 2: Semantic Argument

- Assume $F$ is not valid and $I$ a falsifying interpretation:
  $I \not\models F$
- Apply proof rules.
- If no contradiction reached and no more rules applicable,
  $F$ is invalid.
- If in every branch of proof a contradiction reached,
  $F$ is valid.

# Proof Rules for Semantic Arguments I

$$\frac{I \models \neg F}{I \not\models F} \qquad \frac{I \not\models \neg F}{I \models F}$$

$$\frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \leftarrow \text{and} \qquad \frac{I \not\models F \wedge G}{I \not\models F \quad \mid \quad I \not\models G}$$

$$\nwarrow \text{or}$$

$$\frac{I \models F \vee G}{I \models F \quad \mid \quad I \models G} \qquad \frac{I \not\models F \vee G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}$$

# Proof Rules for Semantic Arguments II

$$\frac{I \models F \rightarrow G}{I \not\models F \quad | \quad I \models G}$$

$$\frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G \quad | \quad I \not\models F \vee G}$$

$$\frac{I \not\models F \leftrightarrow G}{I \models F \wedge \neg G \quad | \quad I \models \neg F \wedge G}$$

$$\frac{\begin{array}{l} I \models F \\ I \not\models F \end{array}}{I \models \bot}$$

Example: Prove

$$F: \; P \land Q \;\rightarrow\; P \lor \neg Q \quad \text{is valid.}$$

Let's assume that $F$ is not valid and that $I$ is a falsifying interpretation.

| | | | | |
|---|---|---|---|---|
| 1. | $I$ | $\not\models$ | $P \land Q \;\rightarrow\; P \lor \neg Q$ | assumption |
| 2. | $I$ | $\models$ | $P \land Q$ | 1 and $\rightarrow$ |
| 3. | $I$ | $\not\models$ | $P \lor \neg Q$ | 1 and $\rightarrow$ |
| 4. | $I$ | $\models$ | $P$ | 2 and $\land$ |
| 5. | $I$ | $\not\models$ | $P$ | 3 and $\lor$ |
| 6. | $I$ | $\models$ | $\perp$ | 4 and 5 are contradictory |

Thus $F$ is valid.

Example: Prove

$$F: (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R) \quad \text{is valid.}$$

Let's assume that $F$ is not valid.

| | | |
|---|---|---|
| 1. | $I \not\models F$ | assumption |
| 2. | $I \models (P \rightarrow Q) \wedge (Q \rightarrow R)$ | 1 and $\rightarrow$ |
| 3. | $I \not\models P \rightarrow R$ | 1 and $\rightarrow$ |
| 4. | $I \models P$ | 3 and $\rightarrow$ |
| 5. | $I \not\models R$ | 3 and $\rightarrow$ |
| 6. | $I \models P \rightarrow Q$ | 2 and $\wedge$ |
| 7. | $I \models Q \rightarrow R$ | 2 and $\wedge$ |

| | | |
|---|---|---|
| 6. | $I \models P \rightarrow Q$ | 2 and $\land$ |
| 7. | $I \models Q \rightarrow R$ | 2 and $\land$ |
| 8a. | $I \not\models P$ | 6 and $\rightarrow$ (case a) |
| 9a. | $I \models \bot$ | 4 and 8 |
| 8b. | $I \models Q$ | 6 and $\rightarrow$ (case b) |
| 9ba. | $I \not\models Q$ | 7 and $\rightarrow$ (subcase ba) |
| 10ba. | $I \models \bot$ | 8b and 9ba |
| 9bb. | $I \models R$ | 7 and $\rightarrow$ (subcase bb) |
| 10bb. | $I \models \bot$ | 5 and 9bb |
| 9b. | $I \models \bot$ | 10ba and 10bb |
| 8. | $I \models \bot$ | 9a and 9b |

Our assumption is contradictory in all cases, so $F$ is valid.

Example 3: Is

$$F: \; P \vee Q \;\rightarrow\; P \wedge Q$$

valid? Assume $F$ is not valid:

| | | |
|---|---|---|
| 1. | $I \not\models P \vee Q \;\rightarrow\; P \wedge Q$ | assumption |
| 2. | $I \models P \vee Q$ | 1 and $\rightarrow$ |
| 3. | $I \not\models P \wedge Q$ | 1 and $\rightarrow$ |
| 4a. | $I \models P$ | 2, $\vee$ (case a) |
| 5aa. | $I \not\models P$ | 3, $\vee$ (subcase aa) |
| 6aa. | $I \models \perp$ | 4a, 5aa |
| 5ab. | $I \not\models Q$ | 3, $\vee$ (subcase ab) |
| 6ab. | ? | |
| 5a. | ? | |

| | | |
|---|---|---|
| 4b. | $I \models Q$ | 2, $\vee$ (case b) |
| 5ba. | $I \not\models P$ | 3, $\vee$ (subcase ba) |
| 6ba. | ? | |
| 5bb. | $I \not\models Q$ | 3, $\vee$ (subcase bb) |
| 6bb. | $I \models \bot$ | 4b, 5bb |
| 5b. | ? | |
| 5. | ? | |

We cannot derive a contradiction in both cases (4a and 4b), so we cannot prove that $F$ is valid. To demonstrate that $F$ is not valid, however, we must find a falsifying interpretation (here are two):

$I_1 : \{P \mapsto \text{true}, Q \mapsto \text{false}\} \qquad I_2 : \{Q \mapsto \text{true}, P \mapsto \text{false}\}$

Note: we have to derive a contradiction in all cases for $F$ to be valid!

# Equivalence

$F_1$ and $F_2$ are <u>equivalent</u> ($F_1 \Leftrightarrow F_2$)

iff for all interpretations $I$, $I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$, show $F_1 \leftrightarrow F_2$ is valid.

$F_1$ <u>entails</u> $F_2$ ($F_1 \Rightarrow F_2$)

iff for all interpretations $I$, $I \models F_1 \rightarrow F_2$

<u>Note</u>: $F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!!

# Normal Forms

1. Negation Normal Form (NNF)

   $\neg, \wedge, \vee$ are the only boolean connectives allowed.

   Negations may occur only in literals of the form $\neg P$.

   To transform $F$ into equivalent $F'$ in NNF, apply the following template equivalences recursively (and left-to-right):

$$\neg\neg F_1 \;\Leftrightarrow\; F_1 \qquad \neg\top \;\Leftrightarrow\; \bot \qquad \neg\bot \;\Leftrightarrow\; \top$$

$$\left.\begin{array}{l} \neg(F_1 \wedge F_2) \;\Leftrightarrow\; \neg F_1 \vee \neg F_2 \\ \neg(F_1 \vee F_2) \;\Leftrightarrow\; \neg F_1 \wedge \neg F_2 \end{array}\right\}\text{De Morgan's Law}$$

$$F_1 \;\rightarrow\; F_2 \;\Leftrightarrow\; \neg F_1 \vee F_2$$

$$F_1 \;\leftrightarrow\; F_2 \;\Leftrightarrow\; (F_1 \;\rightarrow\; F_2) \wedge (F_2 \;\rightarrow\; F_1)$$

   "Complete" syntactic restriction: every $F$ has a corresponding $F'$ in NNF.

Example: Convert

$$F : \neg(P \rightarrow \neg(P \wedge Q))$$

to NNF.

$$\begin{array}{rll} F' & : \neg(\neg P \vee \neg(P \wedge Q)) & \rightarrow \\ F'' & : \neg\neg P \wedge \neg\neg(P \wedge Q) & \text{De Morgan's Law} \\ F''' & : P \wedge P \wedge Q & \neg\neg \end{array}$$

$F'''$ is equivalent to $F$ ($F''' \Leftrightarrow F$) and is in NNF.

2. Disjunctive Normal Form (DNF)

Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert $F$ into equivalent $F'$ in DNF,
transform $F$ into NNF and then
use the following template equivalences (left-to-right):

$$\left.\begin{array}{rcl} (F_1 \vee F_2) \wedge F_3 & \Leftrightarrow & (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \\ F_1 \wedge (F_2 \vee F_3) & \Leftrightarrow & (F_1 \wedge F_2) \vee (F_1 \wedge F_3) \end{array}\right\} \mathit{dist}$$

Note: formulae can grow exponentially as the distributivity laws are applied.

Example: Convert

$$F : \ (Q_1 \vee \neg\neg Q_2) \wedge (\neg R_1 \ \rightarrow \ R_2)$$

into equivalent DNF

$$
\begin{array}{rll}
F' : & (Q_1 \vee Q_2) \wedge (R_1 \vee R_2) & \text{in NNF} \\
F'' : & (Q_1 \wedge (R_1 \vee R_2)) \vee (Q_2 \wedge (R_1 \vee R_2)) & \text{dist} \\
F''' : & (Q_1 \wedge R_1) \vee (Q_1 \wedge R_2) \vee (Q_2 \wedge R_1) \vee (Q_2 \wedge R_2) & \text{dist}
\end{array}
$$

$F'''$ is equivalent to $F$ ($F''' \Leftrightarrow F$) and is in DNF.

3. Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert $F$ into equivalent $F'$ in CNF,
transform $F$ into NNF and then
use the following template equivalences (left-to-right):

$$\begin{array}{rcl} (F_1 \wedge F_2) \vee F_3 & \Leftrightarrow & (F_1 \vee F_3) \wedge (F_2 \vee F_3) \\ F_1 \vee (F_2 \wedge F_3) & \Leftrightarrow & (F_1 \vee F_2) \wedge (F_1 \vee F_3) \end{array}$$

A disjunction of literals is called a clause.

Example: Convert

$$F : P \leftrightarrow (Q \rightarrow R)$$

to an equivalent formula $F'$ in CNF.

First get rid of $\leftrightarrow$ :

$$F_1 : (P \rightarrow (Q \rightarrow R)) \wedge ((Q \rightarrow R) \rightarrow P)$$

Now replace $\rightarrow$ with $\vee$:

$$F_2 : (\neg P \vee (\neg Q \vee R)) \wedge (\neg(\neg Q \vee R) \vee P)$$

Drop unnecessary parentheses and apply De Morgan's Law:

$$F_3 : (\neg P \vee \neg Q \vee R) \wedge ((\neg\neg Q \wedge \neg R) \vee P)$$

Simplify double negation (now in NNF):

$$F_4 : (\neg P \vee \neg Q \vee R) \wedge ((Q \wedge \neg R) \vee P)$$

Distribute disjunction over conjunction (now in CNF):

$$F' : (\neg P \vee \neg Q \vee R) \wedge (Q \vee P) \wedge (\neg R \vee P)$$

# Equisatisfiability

**Definition**

$F$ and $F'$ are *equisatisfiable*, iff

$$F \text{ is satisfiable if and only if } F' \text{ is satisfiable}$$

Every formula is equisatifiable to either $\top$ or $\bot$.

<u>Goal</u>: Decide satisfiability of PL formula $F$

<u>Step 1</u>: Convert $F$ to equisatisfiable formula $F'$ in CNF

<u>Step 2</u>: Decide satisfiability of formula $F'$ in CNF

# Step 1: Convert $F$ to equisatisfiable formula $F'$ in CNF I

There is an *efficient conversion* of $F$ to $F'$ where

- $F'$ is in CNF and
- $F$ and $F'$ are equisatisfiable

Note: efficient means polynomial in the size of $F$.

Basic Idea:

- Introduce a new variable $P_G$ for every subformula $G$ of $F$, unless $G$ is already an atom.

# Step 1: Convert $F$ to equisatisfiable formula $F'$ in CNF II

- For each subformula

  $$G : \; G_1 \circ G_2,$$

  produce a small formula

  $$P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2}.$$

  Here $\circ$ denotes an arbitrary connective ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$); if the connective is $\neg$, $G_1$ should be ignored.

# Step 1: Convert $F$ to equisatisfiable formula $F'$ in CNF III

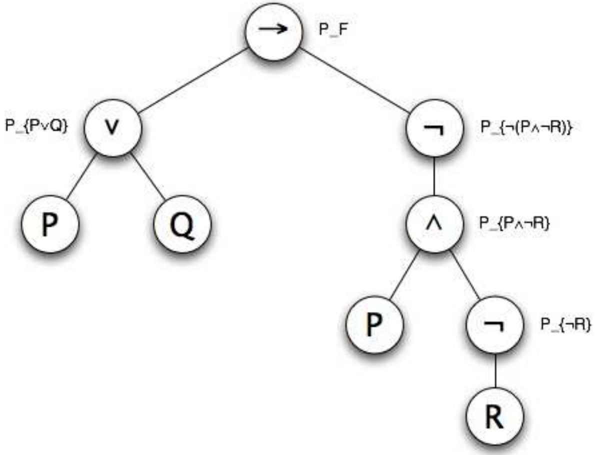



Figure: Parse tree for $F : P \vee Q \ \rightarrow \ \neg(P \wedge \neg R)$

# Step 1: Convert $F$ to equisatisfiable formula $F'$ in CNF IV

- Convert each of these (small) formulae separately to an equivalent CNF formula

$$\mathrm{CNF}(P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2}) \;.$$

Let $S_F$ be the set of all non-atom subformulae $G$ of $F$ (including $F$ itself). The formula

$$P_F \wedge \bigwedge_{G \in S_F} CNF(P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2})$$

is equisatisfiable to $F$. (Why?)

The number of subformulae is linear in the size of $F$.
The time to convert one small formula is constant!

# Example: CNF I

Convert

$$F: \; P \lor Q \;\rightarrow\; P \land \neg R$$

to an equisatisfiable formula in CNF.

Introduce new variables: $P_F$, $P_{P \lor Q}$, $P_{P \land \neg R}$, $P_{\neg R}$.

Create new formulae and convert them to equivalent formulae in CNF separately:

- $F_1 = \text{CNF}(P_F \;\leftrightarrow\; (P_{P \lor Q} \;\rightarrow\; P_{P \land \neg R}))$:

$$(\neg P_F \lor \neg P_{P \lor Q} \lor P_{P \land \neg R}) \land (P_F \lor P_{P \lor Q}) \land (P_F \lor \neg P_{P \land \neg R})$$

- $F_2 = \text{CNF}(P_{P \lor Q} \;\leftrightarrow\; P \lor Q)$:

$$(\neg P_{P \lor Q} \lor P \lor Q) \land (P_{P \lor Q} \lor \neg P) \land (P_{P \lor Q} \lor \neg Q)$$

# Example: CNF II

- $F_3 = \mathrm{CNF}(P_{P \wedge \neg R} \;\leftrightarrow\; P \wedge P_{\neg R})$:

$$(\neg P_{P \wedge \neg R} \vee P) \wedge (\neg P_{P \wedge \neg R} \vee P_{\neg R}) \wedge (P_{P \wedge \neg R} \vee \neg P \vee \neg P_{\neg R})$$

- $F_4 = \mathrm{CNF}(P_{\neg R} \;\leftrightarrow\; \neg R)$:

$$(\neg P_{\neg R} \vee \neg R) \wedge (P_{\neg R} \vee R)$$

$P_F \wedge F_1 \wedge F_2 \wedge F_3 \wedge F_4$ is in CNF and equisatisfiable to $F$.

# Step 2: Decide the satisfiability of PL formula $F'$ in CNF

Boolean Constraint Propagation (BCP)

If a clause contains one literal $\ell$,

| | |
|---|---|
| Set $\ell$ to $\top$: | $\cdots \wedge \overset{\top}{\cancel{\ell}} \wedge \cdots$ |
| Remove all clauses containing $\ell$: | $\cdots \wedge \cancel{(\cdots \vee \ell \vee \cdots)} \wedge \cdots$ |
| Remove $\neg\ell$ in all clauses: | $\cdots \wedge (\cdots \vee \cancel{\neg\ell} \vee \cdots) \wedge \cdots$ |

based on the unit resolution

$$\frac{\ell \qquad \neg\ell \vee C}{C} \leftarrow \text{clause}$$

Pure Literal Propagation (PLP)

If $P$ occurs only positive (without negation), set it to $\top$.
If $P$ occurs only negative set it to $\bot$.
Then do the simplifications as in Boolean Constraint Propagation

# Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Decides the satisfiability of PL formulae in CNF

Decision Procedure DPLL: Given $F$ in CNF

$\texttt{let rec}$ $\textsc{dpll}\ F =$

$\quad \texttt{let}\ F' = \textsc{bcp}\ F\ \texttt{in}$

$\quad \texttt{let}\ F'' = \textsc{plp}\ F'\ \texttt{in}$

$\quad \texttt{if}\ F'' = \top\ \texttt{then true}$

$\quad \texttt{else if}\ F'' = \bot\ \texttt{then false}$

$\quad \texttt{else}$

$\qquad \texttt{let}\ P = \textsc{choose}\ \mathsf{vars}(F'')\ \texttt{in}$

$\qquad (\textsc{dpll}\ F''\{P \mapsto \top\}) \vee (\textsc{dpll}\ F''\{P \mapsto \bot\})$

# Simplification

Simplify according to the template equivalences (left-to-right) [exercise 1.2]

$$
\begin{array}{rclcrclcrcl}
\neg\bot & \Leftrightarrow & \top & \quad & \neg\top & \Leftrightarrow & \bot & \quad & \neg\neg F & \Leftrightarrow & F \\
F \land \top & \Leftrightarrow & F & & F \land \bot & \Leftrightarrow & \bot & & & \cdots & \\
F \lor \top & \Leftrightarrow & \top & & F \lor \bot & \Leftrightarrow & F & & & \cdots &
\end{array}
$$

# Example I

Consider

$$F: \ (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

<u>Branching on $Q$</u>

On the first branch, we have

$$F\{Q \mapsto \top\}: \ (R) \wedge (\neg R) \wedge (P \vee \neg R).$$

By unit resolution,

$$\frac{R \qquad (\neg R)}{\bot},$$

so $F\{Q \mapsto \top\} = \bot \Rightarrow$ false.

# Example II

Recall

$$F: \ (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

On the other branch, we have

$$F\{Q \mapsto \bot\}: \ (\neg P \vee R).$$

Furthermore, by PLP,

$$F\{Q \mapsto \bot, \ R \mapsto \top, \ P \mapsto \bot\} = \top \Rightarrow \text{true}$$

Thus $F$ is satisfiable with satisfying interpretation

$$I: \ \{P \mapsto \text{false}, \ Q \mapsto \text{false}, \ R \mapsto \text{true}\}.$$

# Example

$$F:\ (\neg P \lor Q \lor R) \land (\neg Q \lor R) \land (\neg Q \lor \neg R) \land (P \lor \neg Q \lor \neg R)$$

$$\begin{tikzcd}
 & F \arrow[dl, "Q \mapsto \top"'] \arrow[dr, "Q \mapsto \bot"] & \\
(R) \land (\neg R) \land (P \lor \neg R) \arrow[d, no head] & & (\neg P \lor R) \arrow[d, no head, "R \mapsto \top"] \\
\bot & & \neg P \arrow[d, no head, "P \mapsto \bot"] \\
 & & I:\ \{P \mapsto \text{false},\ Q \mapsto \text{false},\ R \mapsto \text{true}\}
\end{tikzcd}$$