

# CS156: The Calculus of Computation

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## Chapter 2: First-Order Logic (FOL)

# First-Order Logic (FOL)

Also called Predicate Logic or Predicate Calculus

## FOL Syntax

variables  $x, y, z, \dots$

constants  $a, b, c, \dots$

functions  $f, g, h, \dots$

terms variables, constants or

n-ary function applied to n terms as arguments

$a, x, f(a), g(x, b), f(g(x, f(b))); \cancel{f(g(x, f(b, y)))} ??$

predicates  $p, q, r, \dots$

atom  $\top, \perp$ , or an n-ary predicate applied to n terms

literal atom or its negation

$p(f(x), g(x, f(x))), \quad \neg p(f(x), g(x, f(x)))$

Note: 0-ary functions: constants

0-ary predicates (propositional variables):  $P, Q, R, \dots$

## quantifiers

existential quantifier  $\exists x. F[x]$

“there exists an  $x$  such that  $F[x]$ ”

Note: the dot notation ( $\exists x.$ ,  $\forall x.$ ) means the scope of the quantifier should extend as far as possible.

universal quantifier  $\forall x. F[x]$

“for all  $x$ ,  $F[x]$ ”

## FOL formula

literal,

application of logical connectives ( $\neg$ ,  $\vee$ ,  $\wedge$ ,  $\rightarrow$ ,  $\leftrightarrow$ ) to formulae,  
or application of a quantifier to a formula

Example: FOL formula

$$\forall x. p(f(x), x) \rightarrow (\exists y. \underbrace{p(f(g(x, y)), g(x, y))}_G \wedge q(x, f(x)))$$

$\underbrace{\hspace{15em}}_F$

The scope of  $\forall x$  is  $F$ .

The scope of  $\exists y$  is  $G$ .

The formula reads:

“for all  $x$ ,

if  $p(f(x), x)$

then there exists a  $y$  such that  $p(f(g(x, y)), g(x, y))$

and  $q(x, f(x))$ ”

## FOL Semantics

An interpretation  $I : (D_I, \alpha_I)$  consists of:

▶ Domain  $D_I$

non-empty set of values or objects

cardinality  $|D_I|$     deck of cards (finite)

                         integers (countably infinite)

                         reals (uncountably infinite)

▶ Assignment  $\alpha_I$

▶ each variable  $x$  assigned value  $x_I \in D_I$

▶ each n-ary function  $f$  assigned

$$f_I : D_I^n \rightarrow D_I$$

In particular, each constant  $a$  (0-ary function) assigned value  $a_I \in D_I$

▶ each n-ary predicate  $p$  assigned

$$p_I : D_I^n \rightarrow \{\text{true}, \text{false}\}$$

In particular, each propositional variable  $P$  (0-ary predicate) assigned truth value (true, false)

Example:  $F : p(f(x, y), z) \rightarrow p(y, g(z, x))$

Interpretation  $I : (D_I, \alpha_I)$  with

$$D_I = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$\alpha_I : \left\{ \begin{array}{l} f \mapsto +, \quad g \mapsto -, \quad p \mapsto >, \\ x \mapsto 13, \quad y \mapsto 42, \quad z \mapsto 1 \end{array} \right\}$$

Therefore, we can write

$$F_I : 13 + 42 > 1 \rightarrow 42 > 1 - 13.$$

$F$  is true under  $I$ .

## Semantics: Quantifiers

An  $x$ -variant of interpretation  $I : (D_I, \alpha_I)$  is an interpretation  $J : (D_J, \alpha_J)$  such that

- ▶  $D_I = D_J$
- ▶  $\alpha_I[y] = \alpha_J[y]$  for all symbols  $y$ , except possibly  $x$

That is,  $I$  and  $J$  agree on everything except possibly the value of  $x$ .

Denote by  $J : I \triangleleft \{x \mapsto v\}$  the  $x$ -variant of  $I$  in which  $\alpha_J[x] = v$  for some  $v \in D_I$ . Then

- ▶  $I \models \forall x. F$  iff for all  $v \in D_I$ ,  $I \triangleleft \{x \mapsto v\} \models F$
- ▶  $I \models \exists x. F$  iff there exists  $v \in D_I$ , s.t.  $I \triangleleft \{x \mapsto v\} \models F$

Example: Consider

$$F : \exists x. f(x) = g(x)$$

and the interpretation

$$I : (D : \{\circ, \bullet\}, \alpha_I)$$

in which

$$\alpha_I : \{f(\circ) \mapsto \circ, f(\bullet) \mapsto \bullet, g(\circ) \mapsto \bullet, g(\bullet) \mapsto \circ\}.$$

The truth value of  $F$  under  $I$  is false; i.e.,  $I[F] = \text{false}$ .

# Satisfiability and Validity I

$F$  is satisfiable iff there exists  $I$  s.t.  $I \models F$

$F$  is valid iff for all  $I$ ,  $I \models F$

$F$  is valid iff  $\neg F$  is unsatisfiable

Semantic rules: given an interpretation  $I$  with domain  $D_I$ ,

$$\frac{I \models \forall x. F[x]}{I \triangleleft \{x \mapsto v\} \models F[x]} \quad \text{for any } v \in D_I$$

$$\frac{I \not\models \forall x. F[x]}{I \triangleleft \{x \mapsto v\} \not\models F[x]} \quad \text{for a fresh } v \in D_I$$

$$\frac{I \models \exists x. F[x]}{I \triangleleft \{x \mapsto v\} \models F[x]} \quad \text{for a fresh } v \in D_I$$

$$\frac{I \not\models \exists x. F[x]}{I \triangleleft \{x \mapsto v\} \not\models F[x]} \quad \text{for any } v \in D_I$$

## Contradiction rule

A contradiction exists if two variants of the original interpretation  $I$  disagree on the truth value of an  $n$ -ary predicate  $p$  for a given tuple of domain values:

$$\frac{J : I \triangleleft \dots \models p(s_1, \dots, s_n) \\ K : I \triangleleft \dots \not\models p(t_1, \dots, t_n) \quad \text{for } i \in \{1, \dots, n\}, \alpha_J[s_i] = \alpha_K[t_i]}{I \models \perp}$$

Intuition: The variants  $J$  and  $K$  are constructed only through the rules for quantification. Hence, the truth value of  $p$  on the given tuple of domain values is already established by  $I$ . Therefore, the disagreement between  $J$  and  $K$  on the truth value of  $p$  indicates a problem with  $I$ .

Example: Is

$$F : (\forall x. p(x)) \leftrightarrow (\neg \exists x. \neg p(x))$$

valid?

Suppose not. Then there is an  $I$  such that  $I \not\models F$  (assumption).

First case:

- |     |   |                                   |
|-----|---|-----------------------------------|
| 1a. | $I \not\models (\forall x. p(x))$                   |                                   |
|     | $\rightarrow (\neg \exists x. \neg p(x))$           | assumption and $\leftrightarrow$  |
| 2a. | $I \models \forall x. p(x)$                         | 1a and $\rightarrow$              |
| 3a. | $I \not\models \neg \exists x. \neg p(x)$           | 1a and $\rightarrow$              |
| 4a. | $I \models \exists x. \neg p(x)$                    | 3a and $\neg$                     |
| 5a. | $I \triangleleft \{x \mapsto v\} \models \neg p(x)$ | 4a and $\exists, v \in D_I$ fresh |
| 6a. | $I \triangleleft \{x \mapsto v\} \not\models p(x)$  | 5a and $\neg$                     |
| 7a. | $I \triangleleft \{x \mapsto v\} \models p(x)$      | 2a and $\forall$                  |

6a and 7a are contradictory.

## Example (continued):

Second case:

- |     |   |                                   |
|-----|---|-----------------------------------|
| 1b. | $I \not\models (\neg\exists x. \neg p(x))$              |                                   |
|     | $\rightarrow (\forall x. p(x))$                         | assumption and $\leftrightarrow$  |
| 2b. | $I \not\models \forall x. p(x)$                         | 1b and $\rightarrow$              |
| 3b. | $I \models \neg\exists x. \neg p(x)$                    | 1b and $\rightarrow$              |
| 4b. | $I \triangleleft \{x \mapsto v\} \not\models p(x)$      | 2b and $\forall, v \in D_I$ fresh |
| 5b. | $I \not\models \exists x. \neg p(x)$                    | 3b and $\neg$                     |
| 6b. | $I \triangleleft \{x \mapsto v\} \not\models \neg p(x)$ | 5b and $\exists$                  |
| 7b. | $I \triangleleft \{x \mapsto v\} \models p(x)$          | 6b and $\neg$                     |

4b and 7b are contradictory.

Both cases end in contradictions for arbitrary  $I$ . Thus  $F$  is valid.

Example: Prove

$$F : p(a) \rightarrow \exists x. p(x)$$

is valid.

Assume otherwise; i.e.,  $F$  is false under interpretation  $I : (D_I, \alpha_I)$ :

- |    |  |                     |
|----|--|---------------------|
| 1. | $I \not\models F$  | assumption          |
| 2. | $I \models p(a)$   | 1 and $\rightarrow$ |
| 3. | $I \not\models \exists x. p(x)$                              | 1 and $\rightarrow$ |
| 4. | $I \triangleleft \{x \mapsto \alpha_I[a]\} \not\models p(x)$ | 3 and $\exists$     |

2 and 4 are contradictory. Thus,  $F$  is valid.

## Translations of English Sentences (famous theorems) into FOL

- ▶ The length of one side of a triangle is less than the sum of the lengths of the other two sides

$$\forall x, y, z. \text{triangle}(x, y, z) \rightarrow \text{length}(x) < \text{length}(y) + \text{length}(z)$$

- ▶ Fermat's Last Theorem.

$$\forall n. \text{integer}(n) \wedge n > 2$$

$$\rightarrow \forall x, y, z.$$

$$\text{integer}(x) \wedge \text{integer}(y) \wedge \text{integer}(z)$$

$$\wedge x > 0 \wedge y > 0 \wedge z > 0$$

$$\rightarrow \exp(x, n) + \exp(y, n) \neq \exp(z, n)$$

Example: Show that

$$F : (\forall x. p(x, x)) \rightarrow (\exists x. \forall y. p(x, y))$$

is invalid.

Find interpretation  $I$  such that  $F$  is false under  $I$ .

Choose  $D_I = \{0, 1\}$

$p_I = \{(0, 0), (1, 1)\}$  i.e.,  $p_I(0, 0)$  and  $p_I(1, 1)$  are true  
 $p_I(0, 1)$  and  $p_I(1, 0)$  are false

$I[\forall x. p(x, x)] = \text{true}$  and  $I[\exists x. \forall y. p(x, y)] = \text{false}$ .

If we can find a falsifying interpretation for  $F$ , then  $F$  is invalid.

Is  $F : (\forall x. p(x, x)) \rightarrow (\forall x. \exists y. p(x, y))$  valid?

## Substitution

Suppose we want to replace one term with another in a formula;  
e.g., we want to rewrite

$$F : \forall y. (p(x, y) \rightarrow p(y, x))$$

as follows:

$$G : \forall y. (p(a, y) \rightarrow p(y, a)).$$

We call the mapping from  $x$  to  $a$  a substitution denoted as

$$\sigma : \{x \mapsto a\}.$$

We write  $F\sigma$  for the formula  $G$ .

Another convenient notation is  $F[x]$  for a formula containing the variable  $x$  and  $F[a]$  for  $F\sigma$ .

# Substitution

## Definition (Substitution)

A substitution is a mapping from terms to terms; e.g.,

$$\sigma : \{t_1 \mapsto s_1, \dots, t_n \mapsto s_n\}.$$

By  $F\sigma$  we denote the application of  $\sigma$  to formula  $F$ ; i.e., the formula  $F$  where all occurrences of  $t_1, \dots, t_n$  are replaced by  $s_1, \dots, s_n$ .

For a formula named  $F[x]$  we write  $F[t]$  as shorthand for  $F[x]\{x \mapsto t\}$ .

## Renaming

Replace  $x$  in  $\forall x$  by  $x'$  and all free occurrences<sup>1</sup> of  $x$  in the scope  $G[x]$  of  $\forall x$  by  $x'$ :

$$\forall x. G[x] \quad \Leftrightarrow \quad \forall x'. G[x'].$$

Same for  $\exists x$ :

$$\exists x. G[x] \quad \Leftrightarrow \quad \exists x'. G[x'],$$

where  $x'$  is a fresh variable.

Example (renaming):

$$\begin{array}{ccccc} (\forall x. p(x) \rightarrow \exists x. q(x)) \wedge r(x) & & & & \\ \uparrow \forall x & & \uparrow \exists x & & \uparrow \text{free} \end{array}$$

replace by the equivalent formula

$$(\forall y. p(y) \rightarrow \exists z. q(z)) \wedge r(x)$$

---

<sup>1</sup>Note: these occurrences are free in  $G[x]$ , *not* in  $\forall x. G[x]$ .

## Safe Substitution I

Care has to be taken in the presence of quantifiers:

$$F[x] : \exists y. y = Succ(x)$$

What is  $F[y]$ ?

We need to rename bound variables occurring in the substitution:

$$F[x] : \exists y'. y' = Succ(x)$$

Bound variable renaming does not change the models of a formula:

$$(\exists y. y = Succ(x)) \Leftrightarrow (\exists y'. y' = Succ(x))$$

Then under safe substitution

$$F[y] : \exists y'. y' = Succ(y)$$

## Safe Substitution II

Example: Consider the following formula and substitution:

$$F : (\forall x. p(x, y)) \rightarrow q(f(y), x)$$

$$\sigma : \{x \mapsto g(x), y \mapsto f(x), f(y) \mapsto h(x, y)\}$$

Note that the only bound variable in  $F$  is the  $x$  in  $p(x, y)$ . The variables  $x$  and  $y$  are free everywhere else.

What is  $F\sigma$ ? Use safe substitution!

1. Rename the bound  $x$  with a fresh name  $x'$ :

$$F' : (\forall x'. p(x', y)) \rightarrow q(f(y), x)$$

2.  $F\sigma : (\forall x'. p(x', f(x))) \rightarrow q(h(x, y), g(x))$

## Safe Substitution III

Proposition (Substitution of Equivalent Formulae)

$$\sigma : \{F_1 \mapsto G_1, \dots, F_n \mapsto G_n\}$$

s.t. for each  $i$ ,  $F_i \Leftrightarrow G_i$

If  $F\sigma$  is a safe substitution, then  $F \Leftrightarrow F\sigma$ .

## Semantic Tableaux (with Substitution)

We assume that there are infinitely many constant symbols.

The following rules are used for quantifiers:

$$\frac{I \models \forall x. F[x]}{I \models F[t]} \quad \text{for any term } t$$

$$\frac{I \not\models \forall x. F[x]}{I \not\models F[a]} \quad \text{for a fresh constant } a$$

$$\frac{I \models \exists x. F[x]}{I \models F[a]} \quad \text{for a fresh constant } a$$

$$\frac{I \not\models \exists x. F[x]}{I \not\models F[t]} \quad \text{for any term } t$$

The contradiction rule is similar to that of propositional logic:

$$\frac{I \models p(t_1, \dots, t_n) \quad I \not\models p(t_1, \dots, t_n)}{I \models \perp}$$

Example: Show that

$$F : (\exists x. \forall y. p(x, y)) \rightarrow (\forall x. \exists y. p(y, x))$$

is valid.

Assume otherwise.

- |    |     |               |                                 |                           |
|----|-----|---------------|---------------------------------|---------------------------|
| 1. | $I$ | $\not\models$ | $F$                             | assumption                |
| 2. | $I$ | $\models$     | $\exists x. \forall y. p(x, y)$ | 1 and $\rightarrow$       |
| 3. | $I$ | $\not\models$ | $\forall x. \exists y. p(y, x)$ | 1 and $\rightarrow$       |
| 4. | $I$ | $\models$     | $\forall y. p(a, y)$            | 2, $\exists$ ( $a$ fresh) |
| 5. | $I$ | $\not\models$ | $\exists y. p(y, b)$            | 3, $\forall$ ( $b$ fresh) |
| 6. | $I$ | $\models$     | $p(a, b)$                       | 4, $\forall$ ( $t := b$ ) |
| 7. | $I$ | $\not\models$ | $p(a, b)$                       | 5, $\forall$ ( $t := a$ ) |
| 8. | $I$ | $\models$     | $\perp$                         | 6, 7 contradictory        |

Thus, the formula is valid.

Example: Is  $F : (\forall x. p(x, x)) \rightarrow (\exists x. \forall y. p(x, y))$  valid?

Assume  $I$  falsifies  $F$  and apply semantic argument:

1.  $I \not\models F$  assumption
2.  $I \models \forall x. p(x, x)$  1 and  $\rightarrow$
3.  $I \not\models \exists x. \forall y. p(x, y)$  1 and  $\rightarrow$
4.  $I \models p(a_1, a_1)$  2,  $\forall$ ,  $a_1 \in D_I$  fresh
5.  $I \not\models \forall y. p(a_1, y)$  3,  $\exists$
6.  $I \models p(a_1, a_2)$  5,  $\forall$ ,  $a_2 \in D_I$  fresh
7.  $I \models p(a_2, a_2)$  2,  $\forall$
8.  $I \not\models \forall y. p(a_2, y)$  3,  $\exists$
9.  $I \models p(a_2, a_3)$  8,  $\forall$ ,  $a_3 \in D_I$  fresh
- $\vdots$

No contradiction. Falsifying interpretation  $I$ :

$$D_I = \mathbb{N}, \quad p_I(x, y) = \begin{cases} \text{true} & y = x, \\ \text{false} & y = x + 1, \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

# Formula Schemata

## Formula

$$(\forall x. p(x)) \leftrightarrow (\neg \exists x. \neg p(x))$$

## Formula Schema

$$H_1 : (\forall x. F) \leftrightarrow (\neg \exists x. \neg F)$$

↑ place holder

## Formula Schema (with side condition)

$$H_2 : (\forall x. F) \leftrightarrow F \quad \text{provided } x \notin \text{free}(F)$$

## Valid Formula Schema

$H$  is valid iff it is valid for any FOL formula  $F_i$  obeying the side conditions.

Example:  $H_1$  and  $H_2$  are valid.

## Substitution $\sigma$ of $H$

$$\sigma : \{F_1 \mapsto G_1, \dots, F_n \mapsto G_n\}$$

mapping place holders  $F_i$  of  $H$  to FOL formulae  $G_i$ ,  
obeying the side conditions of  $H$

### Proposition (Formula Schema)

If  $H$  is a valid formula schema, and  
 $\sigma$  is a substitution obeying  $H$ 's side conditions,  
then  $H\sigma$  is also valid.

### Example:

$H : (\forall x. F) \leftrightarrow F$  provided  $x \notin \text{free}(F)$  is valid.

$\sigma : \{F \mapsto p(y)\}$  obeys the side condition.

Therefore  $H\sigma : \forall x. p(y) \leftrightarrow p(y)$  is valid.

# Proving Validity of Formula Schemata I

Example: Prove validity of

$$H : (\forall x. F) \leftrightarrow F \quad \text{provided } x \notin \text{free}(F).$$

Proof by contradiction. Consider the two directions of  $\leftrightarrow$ .

► First case

1.  $I \models \forall x. F$  assumption
2.  $I \not\models F$  assumption
3.  $I \models F$  1,  $\forall$ , since  $x \notin \text{free}(F)$
4.  $I \models \perp$  2, 3

## Proving Validity of Formula Schemata II

### ► Second Case

1.  $I \not\models \forall x. F$  assumption
2.  $I \models F$  assumption
3.  $I \models \exists x. \neg F$  1 and  $\neg$
4.  $I \models \neg F$  3,  $\exists$ , since  $x \notin \text{free}(F)$
5.  $I \models \perp$  2, 4

Hence,  $H$  is a valid formula schema.

# Normal Forms

## 1. Negation Normal Forms (NNF)

Apply the additional equivalences (left-to-right)

$$\neg\forall x. F[x] \Leftrightarrow \exists x. \neg F[x]$$

$$\neg\exists x. F[x] \Leftrightarrow \forall x. \neg F[x]$$

when converting PL formulae into NNF.

Example:  $G : \forall x. (\exists y. p(x, y) \wedge p(x, z)) \rightarrow \exists w. p(x, w) .$

1.  $\forall x. (\exists y. p(x, y) \wedge p(x, z)) \rightarrow \exists w. p(x, w)$

2.  $\forall x. \neg(\exists y. p(x, y) \wedge p(x, z)) \vee \exists w. p(x, w)$

$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \vee F_2$$

3.  $\forall x. (\forall y. \neg(p(x, y) \wedge p(x, z))) \vee \exists w. p(x, w)$

$$\neg\exists x. F[x] \Leftrightarrow \forall x. \neg F[x]$$

4.  $G' : \forall x. (\forall y. \neg p(x, y) \vee \neg p(x, z)) \vee \exists w. p(x, w)$

$G'$  in NNF and  $G' \Leftrightarrow G$ .

## 2. Prenex Normal Form (PNF)

All quantifiers appear at the beginning of the formula

$$Q_1 x_1 \cdots Q_n x_n. F[x_1, \cdots, x_n]$$

where  $Q_i \in \{\forall, \exists\}$  and  $F$  is quantifier-free.

Every FOL formula  $F$  can be transformed to formula  $F'$  in PNF  
s.t.  $F' \Leftrightarrow F$ :

- ▶ Write  $F$  in NNF,
- ▶ rename quantified variables to fresh names, and
- ▶ move all quantifiers to the front. Be careful!

Example: Find equivalent PNF of

$$F : \forall x. \neg(\exists y. p(x, y) \wedge p(x, z)) \vee \exists y. p(x, y)$$

↑ to the end of the formula

1. Write  $F$  in NNF

$$F_1 : \forall x. (\forall y. \neg p(x, y) \vee \neg p(x, z)) \vee \exists y. p(x, y)$$

2. Rename quantified variables to fresh names

$$F_2 : \forall x. (\forall y. \neg p(x, y) \vee \neg p(x, z)) \vee \exists w. p(x, w)$$

↑ in the scope of  $\forall x$

3. Remove all quantifiers to produce quantifier-free formula

$$F_3 : \neg p(x, y) \vee \neg p(x, z) \vee p(x, w)$$

4. Add the quantifiers before  $F_3$

$$F_4 : \forall x. \forall y. \exists w. \neg p(x, y) \vee \neg p(x, z) \vee p(x, w)$$

Alternately,

$$F'_4 : \forall x. \exists w. \forall y. \neg p(x, y) \vee \neg p(x, z) \vee p(x, w)$$

Note: In  $F_2$ ,  $\forall y$  is in the scope of  $\forall x$ , therefore the order of quantifiers must be  $\dots \forall x \dots \forall y \dots$

$$F_4 \Leftrightarrow F \text{ and } F'_4 \Leftrightarrow F$$

Note: However,  $G \not\Leftrightarrow F$ :

$$G : \forall y. \exists w. \forall x. \neg p(x, y) \vee \neg p(x, z) \vee p(x, w)$$

What about  $F''_4 : \exists w. \forall x. \forall y. \dots$  ?

# Decidability of FOL

- ▶ FOL is undecidable (Turing & Church)

There does not exist an algorithm for deciding if a FOL formula  $F$  is {valid, satisfiable}; i.e., that always halts and says “yes” if  $F$  is {valid, satisfiable} or “no” if  $F$  is {invalid, unsatisfiable}.

- ▶ FOL is semi-decidable

There is a procedure that always halts and says “yes” if  $F$  is {valid, unsatisfiable}, but may not halt if  $F$  is {invalid, satisfiable}.

On the other hand,

- ▶ PL is decidable

There does exist an algorithm for deciding if a PL formula  $F$  is {valid, satisfiable}; e.g., the truth-table procedure.

# Semantic Argument Method

To show FOL formula  $F$  is valid, assume  $I \not\models F$  and derive a contradiction  $I \models \perp$  in all branches

- ▶ Method is sound

If every branch of a semantic argument proof reaches  $I \models \perp$ , then  $F$  is valid

- ▶ Method is complete

Each valid formula  $F$  has a semantic argument proof in which every branch reaches  $I \models \perp$