

CS156: The Calculus of Computation

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Chapter 7: Quantified Linear Arithmetic

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Example: $\exists x. 2x = y$

For Σ_Q -formula

$$F : \exists x. 2x = y,$$

quantifier-free T_Q -equivalent Σ_Q -formula is

$$G : \top$$

For Σ_Z -formula

$$F : \exists x. 2x = y,$$

there is no quantifier-free T_Z -equivalent Σ_Z -formula.

Let \widehat{T}_Z be T_Z with divisibility predicates $|$.

For $\widehat{\Sigma}_Z$ -formula

$$F : \exists x. 2x = y,$$

a quantifier-free \widehat{T}_Z -equivalent $\widehat{\Sigma}_Z$ -formula is

$$G : 2 \mid y.$$

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Quantifier Elimination (QE)

Algorithm for elimination of all quantifiers of formula F until quantifier-free formula (qff) G that is equivalent to F remains

Note: Could be enough if F is equisatisfiable to G , that is F is satisfiable iff G is satisfiable

A theory T admits quantifier elimination iff

there is an algorithm that given Σ -formula F returns

a quantifier-free Σ -formula G that is T -equivalent to F .

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About QE Algorithm

In developing a QE algorithm for theory T , we need only consider formulae of the form

$$\exists x. F$$

for quantifier-free F .

Example: For Σ -formula

$$G_1 : \exists x. \forall y. \underbrace{\exists z. F_1[x, y, z]}_{F_2[x, y]}$$

$$G_2 : \exists x. \forall y. F_2[x, y]$$

$$G_3 : \exists x. \underbrace{\neg \exists y. \neg F_2[x, y]}_{F_3[x]}$$

$$G_4 : \underbrace{\exists x. \neg F_3[x]}_{F_4}$$

$$G_5 : F_4$$

G_5 is quantifier-free and T -equivalent to G_1

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Quantifier Elimination for $T_{\mathbb{Z}}$

$\Sigma_{\mathbb{Z}}: \{\dots, -2, -1, 0, 1, 2, \dots, -3, -2, 2, 3, \dots, +, -, =, <\}$

Lemma:

Given quantifier-free $\Sigma_{\mathbb{Z}}$ -formula $F[y]$ s.t. $\text{free}(F[y]) = \{y\}$.
 $F[t]$ represents the set of integers

$$S: \{n \in \mathbb{Z} : F[n] \text{ is } T_{\mathbb{Z}}\text{-valid}\}.$$

Either $S \cap \mathbb{Z}^+$ or $\mathbb{Z}^+ \setminus S$ is finite.

Note: \mathbb{Z}^+ is the set of positive integers.

Example: $\Sigma_{\mathbb{Z}}$ -formula $F[y]: \exists x. 2x = y$

S : even integers

$S \cap \mathbb{Z}^+$: positive even integers — infinite

$\mathbb{Z}^+ \setminus S$: positive odd integers — infinite

Therefore, by the lemma, there is no quantifier-free $T_{\mathbb{Z}}$ -formula that is $T_{\mathbb{Z}}$ -equivalent to $F[y]$.

Thus, $T_{\mathbb{Z}}$ does not admit QE.



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Augmented theory $\widehat{T}_{\mathbb{Z}}$

$\widehat{\Sigma}_{\mathbb{Z}}: \Sigma_{\mathbb{Z}}$ with countable number of unary divisibility predicates
 $k \mid \cdot$ for $k \in \mathbb{Z}^+$

Intended interpretations:

$k \mid x$ holds iff k divides x without any remainder

Example:

$$x > 1 \wedge y > 1 \wedge 2 \mid x + y$$

is satisfiable (choose $x = 2, y = 2$).

$$\neg(2 \mid x) \wedge 4 \mid x$$

is not satisfiable.

Axioms of $\widehat{T}_{\mathbb{Z}}$: axioms of $T_{\mathbb{Z}}$ with additional countable set of axioms

$$\forall x. k \mid x \leftrightarrow \exists y. x = ky \quad \text{for } k \in \mathbb{Z}^+$$



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$\widehat{T}_{\mathbb{Z}}$ admits QE (Cooper's method)

Algorithm: Given $\widehat{\Sigma}_{\mathbb{Z}}$ -formula

$$\exists x. F[x],$$

where F is quantifier-free, construct quantifier-free $\widehat{\Sigma}_{\mathbb{Z}}$ -formula that is equivalent to $\exists x. F[x]$.

1. Put $F[x]$ into Negation Normal Form (NNF).
2. Normalize literals: $s < t$, $k \mid t$, or $\neg(k \mid t)$.
3. Put x in $s < t$ on one side: $hx < t$ or $s < hx$.
4. Replace hx with x' without a factor.
5. Replace $F[x']$ by $\bigvee F[j]$ for finitely many j .



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Cooper's Method: Step 1

Put $F[x]$ in Negation Normal Form (NNF) $F_1[x]$, so that $\exists x. F_1[x]$

- ▶ has negations only in literals (only \wedge, \vee)
- ▶ is $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\exists x. F[x]$

Example:

$$\exists x. \neg(x - 6 < z - x \wedge 4 \mid 5x + 1 \rightarrow 3x < y)$$

is equivalent to

$$\exists x. x - 6 < z - x \wedge 4 \mid 5x + 1 \wedge \neg(3x < y)$$



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Cooper's Method: Step 3

Replace (left to right)

$$\begin{aligned}
 s = t &\Leftrightarrow s < t + 1 \wedge t < s + 1 \\
 \neg(s = t) &\Leftrightarrow s < t \vee t < s \\
 \neg(s < t) &\Leftrightarrow t < s + 1
 \end{aligned}$$

The output $\exists x. F_2[x]$ contains only literals of form

$$s < t, \quad k \mid t, \quad \text{or} \quad \neg(k \mid t),$$

where s, t are $\widehat{T}_{\mathbb{Z}}$ -terms and $k \in \mathbb{Z}^+$.

Example:

$$\begin{aligned}
 &\neg(x < y) \wedge \neg(x = y + 3) \\
 &\quad \Downarrow \\
 &y < x + 1 \wedge (x < y + 3 \vee y + 3 < x)
 \end{aligned}$$

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Cooper's Method: Step 3

Collect terms containing x so that literals have the form

$$hx < t, \quad t < hx, \quad k \mid hx + t, \quad \text{or} \quad \neg(k \mid hx + t),$$

where t is a term (does not contain x) and $h, k \in \mathbb{Z}^+$. The output is the formula $\exists x. F_3[x]$, which is $\widehat{T}_{\mathbb{Z}}$ -equivalent to $\exists x. F[x]$.

Example:

$$\begin{array}{cc}
 x + x + y < z + 3z + 2y - 4x & 5 \mid -7x + t \\
 \Downarrow & \Downarrow \\
 6x < 4z + y & 5 \mid 7x - t
 \end{array}$$

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Cooper's Method: Step 4 I

Let

$$\delta' = \text{lcm}\{h : h \text{ is a coefficient of } x \text{ in } F_3[x]\},$$

where lcm is the least common multiple. Multiply atoms in $F_3[x]$ by constants so that δ' is the coefficient of x everywhere:

$$\begin{aligned}
 hx < t &\Leftrightarrow \delta'x < h't && \text{where } h'h = \delta' \\
 t < hx &\Leftrightarrow h't < \delta'x && \text{where } h'h = \delta' \\
 k \mid hx + t &\Leftrightarrow h'k \mid \delta'x + h't && \text{where } h'h = \delta' \\
 \neg(k \mid hx + t) &\Leftrightarrow \neg(h'k \mid \delta'x + h't) && \text{where } h'h = \delta'
 \end{aligned}$$

The result $\exists x. F_3'[x]$, in which all occurrences of x in $F_3[x]$ are in terms $\delta'x$.

Replace $\delta'x$ terms in F_3' with a fresh variable x' to form

$$F_3'' : F_3\{\delta'x \mapsto x'\}$$

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Cooper's Method: Step 4 II

Finally, construct

$$\exists x'. \underbrace{F_3''[x'] \wedge \delta' \mid x'}_{F_4[x']}$$

$\exists x'. F_4[x']$ is equivalent to $\exists x. F[x]$ and each literal of $F_4[x']$ has one of the forms:

- (A) $x' < t$
- (B) $t < x'$
- (C) $k \mid x' + t$
- (D) $\neg(k \mid x' + t)$

where t is a term that does not contain x , and $k \in \mathbb{Z}^+$.

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Cooper's Method: Step 4 III

Example: \widehat{T}_2 -formula

$$\exists x. \underbrace{3x + 1 > y \wedge 2x - 6 < z \wedge 4 \mid 5x + 1}_{F[x]}$$

After step 3:

$$\exists x. \underbrace{2x < z + 6 \wedge y - 1 < 3x \wedge 4 \mid 5x + 1}_{F_3[x]}$$

Collecting coefficients of x (step 4):

$$\delta' = \text{lcm}(2, 3, 5) = 30$$

Multiply when necessary:

$$\exists x. 30x < 15z + 90 \wedge 10y - 10 < 30x \wedge 24 \mid 30x + 6$$

Cooper's Method: Step 4 IV

Multiply when necessary:

$$\exists x. 30x < 15z + 90 \wedge 10y - 10 < 30x \wedge 24 \mid 30x + 6$$

Replacing $30x$ with fresh x' and adding divisibility conjunct:

$$\exists x'. \underbrace{x' < 15z + 90 \wedge 10y - 10 < x' \wedge 24 \mid x' + 6 \wedge 30 \mid x'}_{F_4[x']}$$

$\exists x'. F_4[x']$ is equivalent to $\exists x. F[x]$.

Cooper's Method: Step 5

Construct left infinite projection $F_{-\infty}[x']$ of $F_4[x']$ by

(A) replacing literals $x' < t$ by \top

(B) replacing literals $t < x'$ by \perp

Idea: very small numbers satisfy (A) literals but not (B) literals

Let

$$\delta = \text{lcm} \left\{ \begin{array}{l} k \text{ of (C) literals } k \mid x' + t \\ k \text{ of (D) literals } \neg(k \mid x' + t) \end{array} \right\}$$

and B be the set of terms t appearing in (B) literals of $F_4[x']$.

Construct

$$F_5 : \bigvee_{j=1}^{\delta} F_{-\infty}[j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[t + j]$$

F_5 is quantifier-free and \widehat{T}_2 -equivalent to $\exists x. F[x]$.

Intuition of Step 5 I

Property (Periodicity)

if $m \mid \delta$

then $m \mid n$ iff $m \mid n + \lambda\delta$ for all $\lambda \in \mathbb{Z}$

That is, $m \mid \cdot$ cannot distinguish between $m \mid n$ and $m \mid n + \lambda\delta$.

By the choice of δ (lcm of the k 's) — no \mid literal in F_5 can distinguish between n and $n + \lambda\delta$, for any $\lambda \in \mathbb{Z}$.

$$F_5 : \bigvee_{j=1}^{\delta} F_{-\infty}[j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[t + j]$$

Intuition of Step 5 II

▶ left disjunct $\bigvee_{j=1}^{\delta} F_{-\infty}[j]$:

Contains only | literals

Asserts: no least $n \in \mathbb{Z}$ s.t. $F_4[n]$.

For if there exists n satisfying $F_{-\infty}$,

then every $n - \lambda\delta$, for $\lambda \in \mathbb{Z}^+$, also satisfies $F_{-\infty}$.

▶ right disjunct $\bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[t+j]$:

Asserts: There is least $n \in \mathbb{Z}$ s.t. $F_4[n]$.

For let t^* be the largest t in $(B) = \{\text{largest } t \mid t < x' \text{ in } (B)\}$.

If $n \in \mathbb{Z}$ is s.t. $F_4[n]$, then

$$\exists j(1 \leq j \leq \delta). t^* + j \leq n \wedge F_4[t^* + j]$$

In other words,

if there is a solution,

then one must appear in δ interval to the right of t^*

Example of Step 5 I

$$\exists x. \underbrace{3x + 1 > y \wedge 2x - 6 < z \wedge 4 \mid 5x + 1}_{F[x]}$$

$F[x]$

\downarrow

$$\exists x'. \underbrace{x' < 15z + 90 \wedge 10y - 10 < x' \wedge 24 \mid x' + 6 \wedge 30 \mid x'}_{F_4[x']}$$

$F_4[x']$

By step 5,

$$F_{-\infty}[x] : \top \wedge \perp \wedge 24 \mid x' + 6 \wedge 30 \mid x',$$

which simplifies to \perp .

Example of Step 5 II

Compute

$$\delta = \text{lcm}\{24, 30\} = 120 \quad \text{and} \quad B = \{10y - 10\}.$$

Then replacing x' by $10y - 10 + j$ in $F_4[x']$ produces

$$F_5 : \bigvee_{j=1}^{120} \left[\begin{array}{l} 10y - 10 + j < 15z + 90 \wedge 10y - 10 < 10y - 10 + j \\ \wedge 24 \mid 10y - 10 + j + 6 \wedge 30 \mid 10y - 10 + j \end{array} \right]$$

which simplifies to

$$F_5 : \bigvee_{j=1}^{120} \left[\begin{array}{l} 10y + j < 15z + 100 \wedge 0 < j \\ \wedge 24 \mid 10y + j - 4 \wedge 30 \mid 10y - 10 + j \end{array} \right].$$

F_5 is quantifier-free and \widehat{T}_2 -equivalent to $\exists x. F[x]$.

Cooper's Method: Example I

$$\exists x. \underbrace{(3x + 1 < 10 \vee 7x - 6 > 7) \wedge 2 \mid x}_{F[x]}$$

$F[x]$

Isolate x terms

$$\exists x. (3x < 9 \vee 13 < 7x) \wedge 2 \mid x,$$

so

$$\delta' = \text{lcm}\{3, 7\} = 21.$$

After multiplying coefficients by proper constants,

$$\exists x. (21x < 63 \vee 39 < 21x) \wedge 42 \mid 21x,$$

we replace $21x$ by x' :

$$\exists x'. \underbrace{(x' < 63 \vee 39 < x')}_{F_4[x']} \wedge 42 \mid x' \wedge 21 \mid x'.$$

$F_4[x']$

Cooper's Method: Example II

Then

$$F_{-\infty}[x'] : (\top \vee \perp) \wedge 42 \mid x' \wedge 21 \mid x',$$

or, simplifying,

$$F_{-\infty}[x'] : 42 \mid x' \wedge 21 \mid x'.$$

Finally,

$$\delta = \text{lcm}\{21, 42\} = 42 \quad \text{and} \quad B = \{39\},$$

so F_5 :

$$\bigvee_{j=1}^{42} (42 \mid j \wedge 21 \mid j) \vee \bigvee_{j=1}^{42} ((39 + j < 63 \vee 39 < 39 + j) \wedge 42 \mid 39 + j \wedge 21 \mid 39 + j).$$

Since $42 \mid 42$ and $21 \mid 42$, the left main disjunct simplifies to \top , so that F_5 is \widehat{T}_Z -equivalent to \top . Thus, $\exists x. F[x]$ is \widehat{T}_Z -valid.

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Cooper's Method: Example I

$$\exists x. \underbrace{2x = y}_{F[x]}$$

Rewriting

$$\exists x. \underbrace{y - 1 < 2x \wedge 2x < y + 1}_{F_3[x]}$$

Then

$$\delta' = \text{lcm}\{2, 2\} = 2,$$

so by Step 4

$$\exists x'. \underbrace{y - 1 < x' \wedge x' < y + 1 \wedge 2 \mid x'}_{F_4[x']}$$

$F_{-\infty}$ produces \perp .

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Cooper's Method: Example II

However,

$$\delta = \text{lcm}\{2\} = 2 \quad \text{and} \quad B = \{y - 1\},$$

so

$$F_5 : \bigvee_{j=1}^2 (y - 1 < y - 1 + j \wedge y - 1 + j < y + 1 \wedge 2 \mid y - 1 + j)$$

Simplifying,

$$F_5 : \bigvee_{j=1}^2 (0 < j \wedge j < 2 \wedge 2 \mid y - 1 + j)$$

and then

$$F_5 : 2 \mid y,$$

which is quantifier-free and \widehat{T}_Z -equivalent to $\exists x. F[x]$.

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Improvement: Symmetric Elimination

In step 5, if there are fewer

(A) literals $x' < t$

than

(B) literals $t < x'$,

construct the right infinite projection $F_{+\infty}[x']$ from $F_4[x']$ by replacing

(A) literal $x' < t$ by \perp

than

(B) literal $t < x'$ by \top

Then right elimination.

$$F_5 : \bigvee_{j=1}^{\delta} F_{+\infty}[-j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in A} F_4[t - j].$$

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Improvement: Eliminating Blocks of Quantifiers I

Given

$$\exists x_1. \dots \exists x_n. F[x_1, \dots, x_n]$$

where F quantifier-free.

Eliminating x_n (left elimination) produces

$$G_1 : \exists x_1. \dots \exists x_{n-1}. \bigvee_{j=1}^{\delta} F_{-\infty}[x_1, \dots, x_{n-1}, j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[x_1, \dots, x_{n-1}, t + j]$$

which is equivalent to

$$G_2 : \bigvee_{j=1}^{\delta} \exists x_1. \dots \exists x_{n-1}. F_{-\infty}[x_1, \dots, x_{n-1}, j] \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} \exists x_1. \dots \exists x_{n-1}. F_4[x_1, \dots, x_{n-1}, t + j]$$

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Improvement: Eliminating Blocks of Quantifiers II

Treat j as a free variable and examine only $1 + |B|$ formulae

- ▶ $\exists x_1. \dots \exists x_{n-1}. F_{-\infty}[x_1, \dots, x_{n-1}, j]$
- ▶ $\exists x_1. \dots \exists x_{n-1}. F_4[x_1, \dots, x_{n-1}, t + j]$ for each $t \in B$

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Example I

$$F : \exists y. \exists x. x < -2 \wedge 1 - 5y < x \wedge 1 + y < 13x$$

Since $\delta' = \text{lcm}\{1, 13\} = 13$

$$\exists y. \exists x. 13x < -26 \wedge 13 - 65y < 13x \wedge 1 + y < 13x$$

Then

$$\exists y. \exists x'. x' < -26 \wedge 13 - 65y < x' \wedge 1 + y < x' \wedge 13 \mid x'$$

There is one (A) literal $x' < \dots$ and two (B) literals $\dots < x'$, we use right elimination.

$$F_{+\infty} = \perp \quad \delta = \{13\} = 13 \quad A = \{-26\}$$

$$\exists y. \bigvee_{j=1}^{13} \left[\begin{array}{l} -26 - j < -26 \wedge 13 - 65y < -26 - j \\ \wedge 1 + y < -26 - j \wedge 13 \mid -26 - j \end{array} \right]$$

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Example II

Commute

$$G[j] : \bigvee_{j=1}^{13} \exists y. j > 0 \wedge 39 + j < 65y \wedge y < -27 - j \wedge 13 \mid -26 - j$$

Treating j as free variable (and removing $j > 0$), apply QE to

$$H[j] : \exists y. 39 + j < 65y \wedge y < -27 - j \wedge 13 \mid -26 - j$$

Simplify...

$$H'[j] : \bigvee_{k=1}^{65} (k < -1794 - 66j \wedge 13 \mid -26 - j \wedge 65 \mid 39 + j + k)$$

Replace $H[j]$ with $H'[j]$ in $G[j]$

$$\bigvee_{j=1}^{13} \bigvee_{k=1}^{65} (k < -1794 - 66j \wedge 13 \mid -26 - j \wedge 65 \mid 39 + j + k)$$

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Example III

This qff formula is \widehat{T}_Z -equivalent to F .

Quantifier Elimination over Rationals

$$\Sigma_{\mathbb{Q}} : \{0, 1, +, -, =, \geq\}$$

Recall: we use $>$ instead of \geq , as

$$x \geq y \Leftrightarrow x > y \vee x = y \quad x > y \Leftrightarrow x \geq y \wedge \neg(x = y).$$

Ferrante & Rackoff's Method

Given a $\Sigma_{\mathbb{Q}}$ -formula $\exists x. F[x]$, where $F[x]$ is quantifier-free, generate quantifier-free formula F_4 (four steps) s.t.

$$F_4 \text{ is } \Sigma_{\mathbb{Q}}\text{-equivalent to } \exists x. F[x]$$

by

1. putting $F[x]$ in NNF,
2. replacing negated literals,
3. solving literals s.t. x appears isolated on one side, and
4. taking finite disjunction $\bigvee_t F[t]$.



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Ferrante & Rackoff's Method: Steps 1 and 2

Step 1: Put $F[x]$ in NNF. The result is $\exists x. F_1[x]$.

Step 2: Replace literals (left to right)

$$\neg(s < t) \Leftrightarrow t < s \vee t = s$$

$$\neg(s = t) \Leftrightarrow t < s \vee t > s$$

The result $\exists x. F_2[x]$ does not contain negations.



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Ferrante & Rackoff's Method: Step 3

Solve for x in each atom of $F_2[x]$, e.g.,

$$t_1 < cx + t_2 \quad \Rightarrow \quad \frac{t_1 - t_2}{c} < x$$

where $c \in \mathbb{Z} - \{0\}$.

All atoms in the result $\exists x. F_3[x]$ have form

(A) $x < t$

(B) $t < x$

(C) $x = t$

where t is a term that does not contain x .



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Construct from $F_3[x]$

- ▶ left infinite projection $F_{-\infty}$ by replacing

- (A) atoms $x < t$ by \top
- (B) atoms $t < x$ by \perp
- (C) atoms $x = t$ by \perp

- ▶ right infinite projection $F_{+\infty}$ by replacing

- (A) atoms $x < t$ by \perp
- (B) atoms $t < x$ by \top
- (C) atoms $x = t$ by \perp

Let S be the set of t terms from (A), (B), (C) atoms.

Construct the final

$$F_4: F_{-\infty} \vee F_{+\infty} \vee \bigvee_{s,t \in S} F_3 \left[\frac{s+t}{2} \right],$$

which is T_Q -equivalent to $\exists x. F[x]$.

- ▶ $F_{-\infty}$ captures the case when small $x \in \mathbb{Q}$ satisfy $F_3[x]$

- ▶ $F_{+\infty}$ captures the case when large $x \in \mathbb{Q}$ satisfy $F_3[x]$

- ▶ last disjunct: for $s, t \in S$

if $s \equiv t$, check whether $s \in S$ satisfies $F_3[s]$

if $s \not\equiv t$, in any T_Q -interpretation,

- ▶ $|S| - 1$ pairs $s, t \in S$ are adjacent. For each such pair, (s, t) is an interval in which no other $s' \in S$ lies.
- ▶ Since $\frac{s+t}{2}$ represents the whole interval (s, t) , simply check $F_3[\frac{s+t}{2}]$.

Ferrante & Rackoff's Method: Intuition

Step 4 says that four cases are possible:

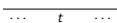
1. There is a left open interval s.t. all elements satisfy $F(x)$.



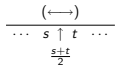
2. There is a right open interval s.t. all elements satisfy $F(x)$.



3. Some term t satisfies $F(x)$.



4. There is an open interval between two s, t terms s.t. every element satisfies $F(x)$.



Correctness of Step 4 I

Theorem

Let F_4 be the formula constructed from $\exists x. F_3[x]$ as in Step 4.

Then $\exists x. F_3[x] \Leftrightarrow F_4$.

Proof:

\Leftarrow If F_4 is true, then $F_{-\infty}$, $F_{+\infty}$ or $F_3[\frac{s+t}{2}]$ is true.

If $F_3[\frac{s+t}{2}]$ is true, then obviously $\exists x. F_3[x]$ is true.

If $F_{-\infty}$ is true, choose some small $x, x < t$ for all $t \in S$.

Then $F_3[x]$ is true.

If $F_{+\infty}$ is true, choose some big $x, x > t$ for all $t \in S$.

Then $F_3[x]$ is true.

Correctness of Step 4 II

\Rightarrow If $I \models \exists x. F_3[x]$ then there is value v such that

$$I \triangleleft \{x \mapsto v\} \models F_3.$$

If $v < \alpha_I[t]$ for all $t \in S$, then $I \models F_{-\infty}$.

If $v > \alpha_I[t]$ for all $t \in S$, then $I \models F_{+\infty}$.

If $v = \alpha_I[t]$ for some $t \in S$, then $I \models F[\frac{t+t}{2}]$.

Otherwise choose largest $s \in S$ with $\alpha_I[s] < v$ and smallest $t \in S$ with $\alpha_I[t] > v$.

Since no atom of F_3 can distinguish between values in interval (s, t) ,

$$I \models F_3[v] \text{ iff } I \models F_3\left[\frac{s+t}{2}\right].$$

Hence, $I \models F[\frac{s+t}{2}]$.

Correctness of Step 4 III

In all cases $I \models F_4$.

Ferrante & Rackoff's Method: Example I

$\Sigma_{\mathbb{Q}}$ -formula

$$\exists x. \underbrace{3x + 1 < 10 \wedge 7x - 6 > 7}_{F_3[x]}$$

Solving for x

$$\exists x. \underbrace{x < 3 \wedge x > \frac{13}{7}}_{F_3[x]}$$

Step 4: $x < 3$ in (A) $\Rightarrow F_{-\infty} = \perp$

$x > \frac{13}{7}$ in (B) $\Rightarrow F_{+\infty} = \perp$

$$F_4 : \bigvee_{s,t \in S} \underbrace{\left(\frac{s+t}{2} < 3 \wedge \frac{s+t}{2} > \frac{13}{7} \right)}_{F_3[\frac{s+t}{2}]}$$

Ferrante & Rackoff's Method: Example II

$S = \{3, \frac{13}{7}\} \Rightarrow$

$$F_3\left[\frac{3+3}{2}\right] = \perp \quad F_3\left[\frac{\frac{13}{7} + \frac{13}{7}}{2}\right] = \perp$$

$$F_3\left[\frac{\frac{13}{7} + 3}{2}\right] : \frac{\frac{13}{7} + 3}{2} < 3 \wedge \frac{\frac{13}{7} + 3}{2} > \frac{13}{7}$$

simplifies to \top .

Thus, $F_4 : \top$ is $T_{\mathbb{Q}}$ -equivalent to $\exists x. F[x]$,
so $\exists x. F[x]$ is $T_{\mathbb{Q}}$ -valid.

Example

$$\exists x. \underbrace{2x > y \wedge 3x < z}_{F[x]}$$

Solving for x

$$\exists x. \underbrace{x > \frac{y}{2} \wedge x < \frac{z}{3}}_{F_3[x]}$$

Step 4: $F_{-\infty} = \perp$, $F_{+\infty} = \perp$, $F_3[\frac{y}{2}] = \perp$ and $F_3[\frac{z}{3}] = \perp$.

$$F_4: \frac{\frac{y}{2} + \frac{z}{3}}{2} > \frac{y}{2} \wedge \frac{\frac{y}{2} + \frac{z}{3}}{2} < \frac{z}{3}$$

which simplifies to:

$$F_4: 2z > 3y$$

F_4 is T_Q -equivalent to $\exists x. F[x]$.