

# CS156: The Calculus of Computation

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## Chapter 4: Induction

### Induction

- ▶ Stepwise induction (for  $T_{PA}$ ,  $T_{cons}$ )
- ▶ Complete induction (for  $T_{PA}$ ,  $T_{cons}$ )  
Theoretically equivalent in power to stepwise induction,  
but sometimes produces more concise proof
- ▶ Well-founded induction  
Generalized complete induction
- ▶ Structural induction  
Over logical formulae

# Stepwise Induction (Peano Arithmetic $T_{PA}$ )

## Axiom schema (induction)

$F[0] \wedge \dots$  base case  
 $(\forall n. F[n] \rightarrow F[n+1]) \dots$  inductive step  
 $\rightarrow \forall x. F[x] \dots$  conclusion

for  $\Sigma_{PA}$ -formulae  $F[x]$  with one free variable  $x$ .

To prove  $\forall x. F[x]$ , the conclusion, i.e.,

$F[x]$  is  $T_{PA}$ -valid for all  $x \in \mathbb{N}$ ,

it suffices to show

- ▶ base case: prove  $F[0]$  is  $T_{PA}$ -valid.
- ▶ inductive step: For arbitrary  $n \in \mathbb{N}$ ,  
assume inductive hypothesis, i.e.,  
 $F[n]$  is  $T_{PA}$ -valid,  
then prove  
 $F[n+1]$  is  $T_{PA}$ -valid.

## Example

Prove:

$$F[n] : 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

for all  $n \in \mathbb{N}$ .

- ▶ *Base case*:  $F[0] : 0 = \frac{0 \cdot 1}{2}$
- ▶ *Inductive step*: Assume  $F[n] : 1 + 2 + \dots + n = \frac{n(n+1)}{2}$ , (IH)  
show

$$\begin{aligned} F[n+1] & : 1 + 2 + \dots + n + (n+1) \\ & = \frac{n(n+1)}{2} + (n+1) && \text{by (IH)} \\ & = \frac{n(n+1) + 2(n+1)}{2} \\ & = \frac{(n+1)(n+2)}{2} \end{aligned}$$

Therefore,

$$\forall n \in \mathbb{N}. 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

### Example:

Theory  $T_{PA}^+$  obtained from  $T_{PA}$  by adding the axioms:

$$\blacktriangleright \forall x. x^0 = 1 \quad (E0)$$

$$\blacktriangleright \forall x, y. x^{y+1} = x^y \cdot x \quad (E1)$$

$$\blacktriangleright \forall x, z. \text{exp}_3(x, 0, z) = z \quad (P0)$$

$$\blacktriangleright \forall x, y, z. \text{exp}_3(x, y + 1, z) = \text{exp}_3(x, y, x \cdot z) \quad (P1)$$

( $\text{exp}_3(x, y, z)$  stands for  $x^y \cdot z$ )

Prove that

$$\boxed{\forall x, y. \text{exp}_3(x, y, 1) = x^y}$$

is  $T_{PA}^+$ -valid.

### First attempt:

$$\forall y \underbrace{[\forall x. \text{exp}_3(x, y, 1) = x^y]}_{F[y]}$$

We chose induction on  $y$ . Why?

### Base case:

$$F[0] : \forall x. \text{exp}_3(x, 0, 1) = x^0$$

For arbitrary  $x \in \mathbb{N}$ ,  $\text{exp}_3(x, 0, 1) = 1$  (P0) and  $x^0 = 1$  (E0).

### Inductive step: Failure.

For arbitrary  $n \in \mathbb{N}$ , we cannot deduce

$$F[n + 1] : \forall x. \text{exp}_3(x, n + 1, 1) = x^{n+1}$$

from the inductive hypothesis

$$F[n] : \forall x. \text{exp}_3(x, n, 1) = x^n$$

## Second attempt: Strengthening

### Strengthened property

$$\boxed{\forall x, y, z. \text{exp}_3(x, y, z) = x^y \cdot z}$$

Implies the desired property (choose  $z = 1$ )

$$\forall x, y. \text{exp}_3(x, y, 1) = x^y$$

### Proof of strengthened property:

Again, induction on  $y$

$$\forall y \underbrace{[\forall x, z. \text{exp}_3(x, y, z) = x^y \cdot z]}_{F[y]}$$

Base case:

$$F[0] : \forall x, z. \text{exp}_3(x, 0, z) = x^0 \cdot z$$

For arbitrary  $x, z \in \mathbb{N}$ ,  $\text{exp}_3(x, 0, z) = z$  (P0) and  $x^0 = 1$  (E0).



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Inductive step: For arbitrary  $n \in \mathbb{N}$

Assume inductive hypothesis

$$F[n] : \forall x, z. \text{exp}_3(x, n, z) = x^n \cdot z \quad (\text{IH})$$

prove

$$F[n+1] : \forall x', z'. \text{exp}_3(x', n+1, z') = x'^{n+1} \cdot z'$$

↑  
note

Consider arbitrary  $x', z' \in \mathbb{N}$ :

$$\text{exp}_3(x', n+1, z') = \text{exp}_3(x', n, x' \cdot z') \quad (\text{P1})$$

$$= x'^n \cdot (x' \cdot z') \quad \text{IH } F[n]; x \mapsto x', z \mapsto x' \cdot z'$$

$$= x'^{n+1} \cdot z' \quad (\text{E1})$$



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# Stepwise Induction (Lists $T_{\text{cons}}$ )

## Axiom schema (induction)

- $(\forall \text{atom } u. F[u]) \wedge \dots$  base case  
 $(\forall u, v. F[v] \rightarrow F[\text{cons}(u, v)]) \dots$  inductive step  
 $\rightarrow \forall x. F[x] \dots$  conclusion

for  $\Sigma_{\text{cons}}$ -formulae  $F[x]$  with one free variable  $x$ .

Note:  $\forall \text{atom } u. F[u]$  stands for  $\forall u. (\text{atom}(u) \rightarrow F[u])$ .

To prove  $\forall x. F[x]$ , i.e.,

$F[x]$  is  $T_{\text{cons}}$ -valid for all lists  $x$ ,

it suffices to show

- ▶ base case: prove  $F[u]$  is  $T_{\text{cons}}$ -valid for arbitrary atom  $u$ .
- ▶ inductive step: For arbitrary lists  $u, v$ ,  
assume inductive hypothesis, i.e.,  
 $F[v]$  is  $T_{\text{cons}}$ -valid,  
then prove  
 $F[\text{cons}(u, v)]$  is  $T_{\text{cons}}$ -valid.

## Example: Theory $T_{\text{cons}}^+$ I

$T_{\text{cons}}$  with axioms

### Concatenating two lists

- ▶  $\forall \text{atom } u. \forall v. \text{concat}(u, v) = \text{cons}(u, v)$  (C0)
- ▶  $\forall u, v, x. \text{concat}(\text{cons}(u, v), x) = \text{cons}(u, \text{concat}(v, x))$  (C1)

## Example: Theory $T_{\text{cons}}^+$ II

Example: for atoms  $a, b, c, d$ ,

$$\begin{aligned} & \text{concat}(\text{cons}(a, \text{cons}(b, c)), d) \\ = & \text{cons}(a, \text{concat}(\text{cons}(b, c), d)) && (C1) \\ = & \text{cons}(a, \text{cons}(b, \text{concat}(c, d))) && (C1) \\ = & \text{cons}(a, \text{cons}(b, \text{cons}(c, d))) && (C0) \end{aligned}$$

$$\begin{aligned} & \text{concat}(\text{cons}(\text{cons}(a, b), c), d) \\ = & \text{cons}(\text{cons}(a, b), \text{concat}(c, d)) && (C1) \\ = & \text{cons}(\text{cons}(a, b), \text{cons}(c, d)) && (C0) \end{aligned}$$

## Example: Theory $T_{\text{cons}}^+$ III

Reversing a list

- ▶  $\forall$  atom  $u$ .  $\text{rvs}(u) = u$  (R0)
- ▶  $\forall x, y$ .  $\text{rvs}(\text{concat}(x, y)) = \text{concat}(\text{rvs}(y), \text{rvs}(x))$  (R1)

Example: for atoms  $a, b, c$ ,

$$\begin{aligned} & \text{rvs}(\text{cons}(a, \text{cons}(b, c))) \\ = & \text{rvs}(\text{concat}(a, \text{concat}(b, c))) && (C0) \\ = & \text{concat}(\text{rvs}(\text{concat}(b, c)), \text{rvs}(a)) && (R1) \\ = & \text{concat}(\text{concat}(\text{rvs}(c), \text{rvs}(b)), \text{rvs}(a)) && (R1) \\ = & \text{concat}(\text{concat}(c, b), a) && (R0) \\ = & \text{concat}(\text{cons}(c, b), a) && (C0) \\ = & \text{cons}(c, \text{concat}(b, a)) && (C1) \\ = & \text{cons}(c, \text{cons}(b, a)) && (C0) \end{aligned}$$

## Example: Theory $T_{\text{cons}}^+$ IV

Deciding if a list is flat;

i.e.,  $\text{flat}(x)$  is true iff every element of list  $x$  is an atom.

$$\blacktriangleright \forall \text{ atom } u. \text{flat}(u) \quad (\text{F0})$$

$$\blacktriangleright \forall u, v. \text{flat}(\text{cons}(u, v)) \leftrightarrow \text{atom}(u) \wedge \text{flat}(v) \quad (\text{F1})$$

Example: for atoms  $a, b, c$ ,

$$\text{flat}(\text{cons}(a, \text{cons}(b, c))) = \text{true}$$

$$\text{flat}(\text{cons}(\text{cons}(a, b), c)) = \text{false}$$

Prove

$$\boxed{\forall x. \underbrace{\text{flat}(x) \rightarrow \text{rvs}(\text{rvs}(x)) = x}_{F[x]}}$$

is  $T_{\text{cons}}^+$ -valid.

Base case: For arbitrary atom  $u$ ,

$$F[u] : \text{flat}(u) \rightarrow \text{rvs}(\text{rvs}(u)) = u$$

by F0 and R0.

Inductive step: For arbitrary lists  $u, v$ , assume the inductive hypothesis

$$F[v] : \text{flat}(v) \rightarrow \text{rvs}(\text{rvs}(v)) = v \quad (\text{IH})$$

and prove

$$F[\text{cons}(u, v)] : \text{flat}(\text{cons}(u, v)) \rightarrow \\ \text{rvs}(\text{rvs}(\text{cons}(u, v))) = \text{cons}(u, v) \quad (*)$$

Case  $\neg \text{atom}(u)$

$$\text{flat}(\text{cons}(u, v)) \Leftrightarrow \text{atom}(u) \wedge \text{flat}(v) \Leftrightarrow \perp$$

by (F1). (\*) holds since its antecedent is  $\perp$ .

Case  $\text{atom}(u)$

$$\text{flat}(\text{cons}(u, v)) \Leftrightarrow \text{atom}(u) \wedge \text{flat}(v) \Leftrightarrow \text{flat}(v)$$

by (F1). Now, show

$$\text{rvs}(\text{rvs}(\text{cons}(u, v))) = \dots = \text{cons}(u, v).$$

Missing steps:

$$\begin{aligned} & \text{rvs}(\text{rvs}(\text{cons}(u, v))) \\ = & \text{rvs}(\text{rvs}(\text{concat}(u, v))) && \text{(C0)} \\ = & \text{rvs}(\text{concat}(\text{rvs}(v), \text{rvs}(u))) && \text{(R1)} \\ = & \text{concat}(\text{rvs}(\text{rvs}(u)), \text{rvs}(\text{rvs}(v))) && \text{(R1)} \\ = & \text{concat}(u, \text{rvs}(\text{rvs}(v))) && \text{(R0)} \\ = & \text{concat}(u, v) && \text{(IH), since flat}(v) \\ = & \text{cons}(u, v) && \text{(C0)} \end{aligned}$$

# Complete Induction (Peano Arithmetic $T_{PA}$ )

Axiom schema (complete induction)

$$(\forall n. \underbrace{(\forall n'. n' < n \rightarrow F[n'])}_{IH} \rightarrow F[n]) \quad \dots \text{ inductive step}$$

$$\rightarrow \forall x. F[x] \quad \dots \text{ conclusion}$$

for  $\Sigma_{PA}$ -formulae  $F[x]$  with one free variable  $x$ .

To prove  $\forall x. F[x]$ , the conclusion i.e.,

$F[x]$  is  $T_{PA}$ -valid for all  $x \in \mathbb{N}$ ,

it suffices to show

- ▶ inductive step: For arbitrary  $n \in \mathbb{N}$ ,  
assume inductive hypothesis, i.e.,  
 $F[n']$  is  $T_{PA}$ -valid for every  $n' \in \mathbb{N}$  such that  $n' < n$ ,  
then prove  
 $F[n]$  is  $T_{PA}$ -valid.

Is base case missing?

No. Base case is implicit in the structure of complete induction.

Note:

- ▶ Complete induction is theoretically equivalent in power to stepwise induction.
- ▶ Complete induction sometimes yields more concise proofs.

Example: Integer division  $\text{quot}(5, 3) = 1$  and  $\text{rem}(5, 3) = 2$

Theory  $T_{PA}^*$  obtained from  $T_{PA}$  by adding the axioms:

- ▶  $\forall x, y. x < y \rightarrow \text{quot}(x, y) = 0$  (Q0)
- ▶  $\forall x, y. y > 0 \rightarrow \text{quot}(x + y, y) = \text{quot}(x, y) + 1$  (Q1)
- ▶  $\forall x, y. x < y \rightarrow \text{rem}(x, y) = x$  (R0)
- ▶  $\forall x, y. y > 0 \rightarrow \text{rem}(x + y, y) = \text{rem}(x, y)$  (R1)

Prove

$$(1) \forall x, y. y > 0 \rightarrow \text{rem}(x, y) < y$$

$$(2) \forall x, y. y > 0 \rightarrow x = y \cdot \text{quot}(x, y) + \text{rem}(x, y)$$

Best proved by complete induction.

Proof of (1)

$$\forall x. \underbrace{\forall y. y > 0 \rightarrow \text{rem}(x, y) < y}_{F[x]}$$

Consider an arbitrary natural number  $x$ .

Assume the inductive hypothesis

$$\forall x'. x' < x \rightarrow \underbrace{\forall y'. y' > 0 \rightarrow \text{rem}(x', y') < y'}_{F[x']} \quad (\text{IH})$$

Prove  $F[x] : \forall y. y > 0 \rightarrow \text{rem}(x, y) < y$ .

Let  $y$  be an arbitrary positive integer

Case  $x < y$ :

$$\begin{aligned} \text{rem}(x, y) &= x && \text{by (R0)} \\ &< y && \text{case} \end{aligned}$$

Case  $\neg(x < y)$ :

Then there is natural number  $n$ ,  $n < x$  s.t.  $x = n + y$

$$\begin{aligned} \text{rem}(x, y) &= \text{rem}(n + y, y) && x = n + y \\ &= \text{rem}(n, y) && (\text{R1}) \\ &< y && \text{IH } (x' \mapsto n, y' \mapsto y) \\ &&& \text{since } n < x \text{ and } y > 0 \end{aligned}$$

## Well-founded Induction I

A binary predicate  $\prec$  over a set  $S$  is a well-founded relation iff there does not exist an infinite decreasing sequence

$$s_1 \succ s_2 \succ s_3 \succ \dots \text{ where } s_i \in S$$

Note: where  $s \prec t$  iff  $t \succ s$

Examples:

- ▶  $<$  is well-founded over the natural numbers.  
Any sequence of natural numbers decreasing according to  $<$  is finite:

$$1023 > 39 > 30 > 29 > 8 > 3 > 0.$$

- ▶  $<$  is not well-founded over the rationals in  $[0, 1]$ .

$$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \dots$$

is an infinite decreasing sequence.

## Well-founded Induction II

- ▶  $<$  is not well-founded over the integers:

$$7200 > \dots > 217 > \dots > 0 > \dots > -17 > \dots$$

- ▶ The strict sublist relation  $\prec_c$  is well-founded over the set of all lists.
- ▶ The relation

$$F \prec G \text{ iff } F \text{ is a strict subformula of } G$$

is well-founded over the set of formulae.

## Well-founded Induction Principle

For theory  $T$  and well-founded relation  $\prec$ ,  
the axiom schema (well-founded induction)

$$(\forall n. (\forall n'. n' \prec n \rightarrow F[n']) \rightarrow F[n]) \rightarrow \forall x. F[x]$$

for  $\Sigma$ -formulae  $F[x]$  with one free variable  $x$ .

To prove  $\forall x. F[x]$ , i.e.,

$F[x]$  is  $T$ -valid for every  $x$ ,  
it suffices to show

- ▶ inductive step: For arbitrary  $n$ ,  
assume inductive hypothesis, i.e.,  
 $F[n']$  is  $T$ -valid for every  $n'$ , such that  $n' \prec n$   
then prove  
 $F[n]$  is  $T$ -valid.

Complete induction in  $T_{PA}$  is a specific instance of well-founded induction, where the well-founded relation  $\prec$  is  $<$ .

## Lexicographic Relation

Given pairs  $(S_i, \prec_i)$  of sets  $S_i$  and well-founded relations  $\prec_i$

$$(S_1, \prec_1), \dots, (S_m, \prec_m)$$

Construct

$$S = S_1 \times \dots \times S_m;$$

i.e., the set of  $m$ -tuples  $(s_1, \dots, s_m)$  where each  $s_i \in S_i$ .

Define lexicographic relation  $\prec$  over  $S$  as

$$\underbrace{(s_1, \dots, s_m)}_s \prec \underbrace{(t_1, \dots, t_m)}_t \Leftrightarrow \bigvee_{i=1}^m \left( s_i \prec_i t_i \wedge \bigwedge_{j=1}^{i-1} s_j = t_j \right)$$

for  $s_i, t_i \in S_i$ .

- If  $(S_1, \prec_1), \dots, (S_m, \prec_m)$  are well-founded, so is  $(S, \prec)$ .

Example:  $S = \{A, \dots, Z\}$ ,  $m = 3$ ,  $CAT \prec DOG$ ,  $DOG \prec DRY$ ,  
 $DOG \prec DOT$ .

Example: For the set  $\mathbb{N}^3$  of triples of natural numbers with the lexicographic relation  $\prec$ ,

$$(5, 2, 17) \prec (5, 4, 3)$$

### Lexicographic well-founded induction principle

For theory  $T$  and well-founded lexicographic relation  $\prec$ ,

$$(\forall \bar{n}. (\forall \bar{n}'. \bar{n}' \prec \bar{n} \rightarrow F[\bar{n}']) \rightarrow F[\bar{n}]) \rightarrow \forall \bar{x}. F[\bar{x}]$$

for  $\Sigma_T$ -formula  $F[\bar{x}]$  with free variables  $\bar{x}$ , is  $T$ -valid.

Same as regular well-founded induction, just

$$\begin{aligned} n &\Rightarrow \text{tuple } \bar{n} = (n_1, \dots, n_m) & x &\Rightarrow \text{tuple } \bar{x} = (x_1, \dots, x_m) \\ n' &\Rightarrow \text{tuple } \bar{n}' = (n'_1, \dots, n'_m) \end{aligned}$$

### Example: Puzzle

Bag of red, yellow, and blue chips

If one chip remains in the bag – remove it (empty bag – the process terminates)

Otherwise, remove two chips at random:

1. If one of the two is red –  
don't put any chips in the bag
2. If both are yellow –  
put one yellow and five blue chips
3. If one of the two is blue and the other not red –  
put ten red chips

Does this process terminate?

Proof: Consider

- ▶ Set  $S : \mathbb{N}^3$  of triples of natural numbers and

- ▶ Well-founded lexicographic relation  $<_3$  for such triples, e.g.

$$(11, 13, 3) \not<_3 (11, 9, 104) \quad (11, 9, 104) <_3 (11, 13, 3)$$

Let  $y, b, r$  be the yellow, blue, and red chips in the bag before a move.

Let  $y', b', r'$  be the yellow, blue, and red chips in the bag after a move.

Show

$$(y', b', r') <_3 (y, b, r)$$

for each possible case. Since  $<_3$  well-founded relation

$\Rightarrow$  only finite decreasing sequences  $\Rightarrow$  process must terminate

1. If one of the two removed chips is red – do not put any chips in the bag

$$\left. \begin{array}{l} (y - 1, b, r - 1) \\ (y, b - 1, r - 1) \\ (y, b, r - 2) \end{array} \right\} <_3 (y, b, r)$$

2. If both are yellow – put one yellow and five blue

$$(y - 1, b + 5, r) <_3 (y, b, r)$$

3. If one is blue and the other not red – put ten red

$$\left. \begin{array}{l} (y - 1, b - 1, r + 10) \\ (y, b - 2, r + 10) \end{array} \right\} <_3 (y, b, r)$$

## Example: Ackermann function

Theory  $T_{\mathbb{N}}^{ack}$  is the theory of Presburger arithmetic  $T_{\mathbb{N}}$  (for natural numbers) augmented with

### Ackermann axioms:

- ▶  $\forall y. ack(0, y) = y + 1$  (L0)
- ▶  $\forall x. ack(x + 1, 0) = ack(x, 1)$  (R0)
- ▶  $\forall x, y. ack(x + 1, y + 1) = ack(x, ack(x + 1, y))$  (S)

Ackermann function grows quickly:

$$ack(0, 0) = 1$$

$$ack(1, 1) = 3$$

$$ack(2, 2) = 7$$

$$ack(3, 3) = 61$$

$$ack(4, 4) = 2^{2^{2^{2^{16}}}} - 3$$

### Proof of termination

Let  $<_2$  be the lexicographic extension of  $<$  to pairs of natural numbers.

- (L0)  $\forall y. ack(0, y) = y + 1$   
does not involve recursive call
- (R0)  $\forall x. ack(x + 1, 0) = ack(x, 1)$   
 $(x + 1, 0) >_2 (x, 1)$
- (S)  $\forall x, y. ack(x + 1, y + 1) = ack(x, ack(x + 1, y))$   
 $(x + 1, y + 1) >_2 (x + 1, y)$   
 $(x + 1, y + 1) >_2 (x, ack(x + 1, y))$

No infinite recursive calls  $\Rightarrow$  the recursive computation of  $ack(x, y)$  terminates for all pairs of natural numbers.

## Proof of property

Use well-founded induction over  $<_2$  to prove

$$\forall x, y. \text{ack}(x, y) > y$$

is  $T_{\mathbb{N}}^{\text{ack}}$  valid.

Consider arbitrary natural numbers  $x, y$ .

Assume the inductive hypothesis

$$\forall x', y'. \underbrace{(x', y') <_2 (x, y) \rightarrow \text{ack}(x', y') > y'}_{F[x', y']} \quad (\text{IH})$$

Show

$$F[x, y] : \text{ack}(x, y) > y.$$

Case  $x = 0$ :

$$\text{ack}(0, y) = y + 1 > y \quad \text{by (L0)}$$

Case  $x > 0 \wedge y = 0$ :

$$\text{ack}(x, 0) = \text{ack}(x - 1, 1) \quad \text{by (R0)}$$

Since

$$\underbrace{(x - 1)}_{x'}, \underbrace{(1)}_{y'} <_2 (x, y)$$

Then

$$\text{ack}(x - 1, 1) > 1 \quad \text{by (IH) } (x' \mapsto x - 1, y' \mapsto 1)$$

Thus

$$\text{ack}(x, 0) = \text{ack}(x - 1, 1) > 1 > 0$$

Case  $x > 0 \wedge y > 0$ :

$$ack(x, y) = ack(x - 1, ack(x, y - 1)) \quad \text{by (S)} \quad (1)$$

Since

$$\underbrace{(x - 1)}_{x'}, \underbrace{ack(x, y - 1)}_{y'} <_2 (x, y)$$

Then

$$ack(x - 1, ack(x, y - 1)) > ack(x, y - 1) \quad (2)$$

by (IH) ( $x' \mapsto x - 1, y' \mapsto ack(x, y - 1)$ ).

Furthermore, since

$$\underbrace{(x)}_{x'}, \underbrace{(y - 1)}_{y'} <_2 (x, y)$$

then

$$ack(x, y - 1) > y - 1 \quad (3)$$

By (1)–(3), we have

$$ack(x, y) \stackrel{(1)}{=} ack(x - 1, ack(x, y - 1)) \stackrel{(2)}{>} ack(x, y - 1) \stackrel{(3)}{>} y - 1$$

Hence

$$ack(x, y) > (y - 1) + 1 = y$$

# Structural Induction

How do we prove properties about logical formulae themselves?

## Structural induction principle

To prove a desired property of formulae,

inductive step: Assume the inductive hypothesis, that for arbitrary formula  $F$ , the desired property holds for every strict subformula  $G$  of  $F$ .

Then prove that  $F$  has the property.

Since atoms do not have strict subformulae, they are treated as base cases.

Note: “strict subformula relation” is well-founded

Example: Prove that

Every propositional formula  $F$  is equivalent to a propositional formula  $F'$  constructed with only  $\top$ ,  $\vee$ ,  $\neg$  (and propositional variables)

Base cases:

$$F : \top \Rightarrow F' : \top$$

$$F : \perp \Rightarrow F' : \neg\top$$

$$F : P \Rightarrow F' : P \quad \text{for propositional variable } P$$

Inductive step:

Assume as the inductive hypothesis that  $G, G_1, G_2$  are equivalent to  $G', G'_1, G'_2$  constructed only from  $\top, \vee, \neg$  (and propositional variables).

$$F : \neg G \quad \Rightarrow \quad F' : \neg G'$$

$$F : G_1 \vee G_2 \quad \Rightarrow \quad F' : G'_1 \vee G'_2$$

$$F : G_1 \wedge G_2 \quad \Rightarrow \quad F' : \neg(\neg G'_1 \vee \neg G'_2)$$

$$F : G_1 \rightarrow G_2 \quad \Rightarrow \quad F' : \neg G'_1 \vee G'_2$$

$$F : G_1 \leftrightarrow G_2 \quad \Rightarrow \quad (G'_1 \rightarrow G'_2) \wedge (G'_2 \rightarrow G'_1) \Rightarrow F' : \dots$$

Each  $F'$  is equivalent to  $F$  and is constructed only by  $\top, \vee, \neg$  by the inductive hypothesis.