# CS156: The Calculus of Computation

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Chapter 7: Quantified Linear Arithmetic



## Quantifier Elimination (QE)

Algorithm for elimination of all quantifiers of formula $F$ until quantifier-free formula (qff) $G$ that is equivalent to $F$ remains

Note: Could be enough if $F$ is equisatisfiable to $G$, that is $F$ is satisfiable iff $G$ is satisfiable

A theory $T$ admits quantifier elimination iff

there is an algorithm that given $\Sigma$-formula $F$ returns

a quantifier-free $\Sigma$-formula $G$ that is $T$-equivalent to $F$.



## Example: $\exists x.\ 2x = y$

For $\Sigma_{\mathbb{Q}}$-formula

$F: \ \exists x.\ 2x = y,$

quantifier-free $T_{\mathbb{Q}}$-equivalent $\Sigma_{\mathbb{Q}}$-formula is

$G: \ \top$

For $\Sigma_{\mathbb{Z}}$-formula

$F: \ \exists x.\ 2x = y,$

there is no quantifier-free $T_{\mathbb{Z}}$-equivalent $\Sigma_{\mathbb{Z}}$-formula.

Let $\widehat{T_{\mathbb{Z}}}$ be $T_{\mathbb{Z}}$ with divisibility predicates $|$.
For $\widehat{\Sigma_{\mathbb{Z}}}$-formula

$F: \ \exists x.\ 2x = y,$

a quantifier-free $\widehat{T_{\mathbb{Z}}}$-equivalent $\widehat{\Sigma_{\mathbb{Z}}}$-formula is

$G: \ 2 \mid y.$



## About QE Algorithm

In developing a QE algorithm for theory $T$, we need only consider formulae of the form

$\exists x.\ F$

for quantifier-free $F$.

Example: For $\Sigma$-formula

$$
\begin{aligned}
G_1 &: \ \exists x.\ \forall y.\ \underbrace{\exists z.\ F_1[x,y,z]}_{F_2[x,y]} \\
G_2 &: \ \exists x.\ \forall y.\ F_2[x,y] \\
G_3 &: \ \exists x.\ \neg \underbrace{\exists y.\ \neg F_2[x,y]}_{F_3[x]} \\
G_4 &: \ \underbrace{\exists x.\ \neg F_3[x]}_{F_4} \\
G_5 &: \ F_4
\end{aligned}
$$

$G_5$ is quantifier-free and $T$-equivalent to $G_1$

## Quantifier Elimination for $T_{\mathbb{Z}}$

$\Sigma_{\mathbb{Z}}: \{\ldots, -2, -1, 0, 1, 2, \ldots, -3\cdot, -2\cdot, 2\cdot, 3\cdot, \ldots, +, -, =, <\}$

Lemma:

Given quantifier-free $\Sigma_{\mathbb{Z}}$-formula $F[y]$ s.t. $\text{free}(F[y]) = \{y\}$.
$S$ represents the set of integers

$$S : \{n \in \mathbb{Z} \ : \ F[n] \text{ is } T_{\mathbb{Z}}\text{-valid}\} \ .$$

Either $S \cap \mathbb{Z}^+$ or $\mathbb{Z}^+ \setminus S$ is finite.
Note: $\mathbb{Z}^+$ is the set of positive integers.

Example: $\Sigma_{\mathbb{Z}}$-formula $F[y] : \exists x.\ 2x = y$

$S$: even integers

$S \cap \mathbb{Z}^+$: positive even integers — infinite
$\mathbb{Z}^+ \setminus S$: positive odd integers — infinite
Therefore, by the lemma, there is no quantifier-free $T_{\mathbb{Z}}$-formula that is $T_{\mathbb{Z}}$-equivalent to $F[y]$.
Thus, $T_{\mathbb{Z}}$ does not admit QE.



## Augmented theory $\widehat{T_{\mathbb{Z}}}$

$\widehat{\Sigma_{\mathbb{Z}}}$: $\Sigma_{\mathbb{Z}}$ with countable number of unary divisibility predicates

$k \mid \cdot$ for $k \in \mathbb{Z}^+$

Intended interpretations:

$k \mid x$ holds iff $k$ divides $x$ without any remainder

Example:

$x > 1 \ \wedge \ y > 1 \ \wedge \ 2 \mid x + y$

is satisfiable (choose $x = 2, y = 2$).

$\neg(2 \mid x) \ \wedge \ 4 \mid x$

is not satisfiable.

Axioms of $\widehat{T_{\mathbb{Z}}}$: axioms of $T_{\mathbb{Z}}$ with additional countable set of axioms

$$\forall x.\ k \mid x \ \leftrightarrow \ \exists y.\ x = ky \quad \text{for } k \in \mathbb{Z}^+$$



## $\widehat{T_{\mathbb{Z}}}$ admits QE (Cooper's method)

Algorithm: Given $\widehat{\Sigma_{\mathbb{Z}}}$-formula

$$\exists x.\ F[x] \ ,$$

where $F$ is quantifier-free, construct quantifier-free $\widehat{\Sigma_{\mathbb{Z}}}$-formula that is equivalent to $\exists x.\ F[x]$.

1. Put F[x] into Negation Normal Form (NNF).
2. Normalize literals: $s < t$, $k|t$, or $\neg(k|t)$.
3. Put $x$ in $s < t$ on one side: $hx < t$ or $s < hx$.
4. Replace $hx$ with $x'$ without a factor.
5. Replace $F[x']$ by $\bigvee F[j]$ for finitely many $j$.



## Cooper's Method: Step 1

Put $F[x]$ in Negation Normal Form (NNF) $F_1[x]$, so that $\exists x.\ F_1[x]$

- has negations only in literals (only $\wedge$, $\vee$)
- is $\widehat{T_{\mathbb{Z}}}$-equivalent to $\exists x.\ F[x]$

Example:

$$\exists x.\ \neg(x - 6 < z - x \ \wedge \ 4 \mid 5x + 1 \ \rightarrow \ 3x < y)$$

is equivalent to

$$\exists x.\ x - 6 < z - x \ \wedge \ 4 \mid 5x + 1 \ \wedge \ \neg(3x < y)$$

Note:

$$\neg(A \wedge B \rightarrow C) \Leftrightarrow (A \wedge B \wedge \neg C)$$

## Cooper's Method: Step 2

Replace (left to right)

$$\begin{array}{rcl} s = t & \Leftrightarrow & s < t+1 \ \wedge\ t < s+1 \\ \neg(s = t) & \Leftrightarrow & s < t \ \vee\ t < s \\ \neg(s < t) & \Leftrightarrow & t < s+1 \end{array}$$

The output $\exists x.\ F_2[x]$ contains only literals of form

$$s < t\ , \quad k \mid t\ , \quad \text{or} \quad \neg(k \mid t)\ ,$$

where $s$, $t$ are $\widehat{T_{\mathbb{Z}}}$-terms and $k \in \mathbb{Z}^+$.

Example:

$$\neg(x < y)\ \wedge\ \neg(x = y+3)$$
$$\Downarrow$$
$$y < x+1\ \wedge\ (x < y+3\ \vee\ y+3 < x)$$



## Cooper's Method: Step 3

Collect terms containing $x$ so that literals have the form

$$hx < t\ , \quad t < hx\ , \quad k \mid hx + t\ , \quad \text{or} \quad \neg(k \mid hx+t)\ ,$$

where $t$ is a term (does not contain $x$) and $h, k \in \mathbb{Z}^+$. The output is the formula $\exists x.\ F_3[x]$, which is $\widehat{T_{\mathbb{Z}}}$-equivalent to $\exists x.\ F[x]$.

Example:

$$x + x + y < z + 3z + 2y - 4x \qquad 5 \mid -7x + t$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$6x < 4z + y \qquad 5 \mid 7x - t$$



## Cooper's Method: Step 4 I

Let

$$\delta' = \text{lcm}\{h\ :\ h \text{ is a coefficient of } x \text{ in } F_3[x]\}\ ,$$

where lcm is the least common multiple. Multiply atoms in $F_3[x]$ by constants so that $\delta'$ is the coefficient of $x$ everywhere:

$$\begin{array}{rclll} hx < t & \Leftrightarrow & \delta' x < h't & \text{where} & h'h = \delta' \\ t < hx & \Leftrightarrow & h't < \delta' x & \text{where} & h'h = \delta' \\ k \mid hx + t & \Leftrightarrow & h'k \mid \delta' x + h't & \text{where} & h'h = \delta' \\ \neg(k \mid hx + t) & \Leftrightarrow & \neg(h'k \mid \delta' x + h't) & \text{where} & h'h = \delta' \end{array}$$

The result $\exists x.\ F_3'[x]$, in which all occurrences of $x$ in $F_3'[x]$ are in terms $\delta' x$.

Replace $\delta' x$ terms in $F_3'$ with a fresh variable $x'$ to form

$$F_3''\ :\ F_3\{\delta' x \mapsto x'\}$$



## Cooper's Method: Step 4 II

Finally, construct

$$\exists x'.\ \underbrace{F_3''[x']\ \wedge\ \delta' \mid x'}_{F_4[x']}$$

$\exists x'.F_4[x']$ is equivalent to $\exists x.\ F[x]$ and each literal of $F_4[x']$ has one of the forms:

(A) $x' < t$
(B) $t < x'$
(C) $k \mid x' + t$
(D) $\neg(k \mid x' + t)$

where $t$ is a term that does not contain $x'$, and $k \in \mathbb{Z}^+$.

## Cooper's Method: Step 4 III

Example: $\widehat{T_{\mathbb{Z}}}$-formula

$$\exists x.\ \underbrace{3x + 1 > y\ \wedge\ 2x - 6 < z\ \wedge\ 4 \mid 5x + 1}_{F[x]}$$

After step 3:

$$\exists x.\ \underbrace{2x < z + 6\ \wedge\ y - 1 < 3x\ \wedge\ 4 \mid 5x + 1}_{F_3[x]}$$

Collecting coefficients of $x$ (step 4):

$$\delta' = \text{lcm}(2, 3, 5) = 30$$

Multiply when necessary:

$$\exists x.\ 30x < 15z + 90\ \wedge\ 10y - 10 < 30x\ \wedge\ 24 \mid 30x + 6$$



## Cooper's Method: Step 4 IV

Multiply when necessary:

$$\exists x.\ 30x < 15z + 90\ \wedge\ 10y - 10 < 30x\ \wedge\ 24 \mid 30x + 6$$

Replacing $30x$ with fresh $x'$ and adding divisibility conjunct:

$$\exists x'.\ \underbrace{x' < 15z + 90\ \wedge\ 10y - 10 < x'\ \wedge\ 24 \mid x' + 6\ \wedge\ 30 \mid x'}_{F_4[x']}$$

$\exists x'.\ F_4[x']$ is equivalent to $\exists x.\ F[x]$.



## Cooper's Method: Step 5

Construct left infinite projection $F_{-\infty}[x']$ of $F_4[x']$ by

(A) replacing literals $x' < t$ by $\top$

(B) replacing literals $t < x'$ by $\bot$

Idea: very small numbers satisfy (A) literals but not (B) literals

Let

$$\delta = \text{lcm}\left\{ \begin{array}{l} k \text{ of (C) literals } k \mid x' + t \\ k \text{ of (D) literals } \neg(k \mid x' + t) \end{array} \right\}$$

and $B$ be the set of terms $t$ appearing in (B) literals of $F_4[x']$.

Construct

$$F_5 : \bigvee_{j=1}^{\delta} F_{-\infty}[j]\ \vee\ \bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[t + j]\ .$$

$F_5$ is quantifier-free and $\widehat{T_{\mathbb{Z}}}$-equivalent to $\exists x.\ F[x]$.



## Intuition of Step 5 I

Property (Periodicity)

if $m \mid \delta$

then $m \mid n$ iff $m \mid n + \lambda\delta$ for all $\lambda \in \mathbb{Z}$

That is, $m \mid \cdot$ cannot distinguish between $m \mid n$ and $m \mid n + \lambda\delta$.

By the choice of $\delta$ (lcm of the $k$'s) — no $\mid$ literal in $F_5$ can distinguish between $n$ and $n + \lambda\delta$, for any $\lambda \in \mathbb{Z}$.

$$F_5 : \bigvee_{j=1}^{\delta} F_{-\infty}[j]\ \vee\ \bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[t + j]$$

## Intuition of Step 5 II

- left disjunct $\bigvee_{j=1}^{\delta} F_{-\infty}[j]$ :

  Contains only | literals
  Asserts: no least $n \in \mathbb{Z}$ s.t. $F_4[n]$.

  For if there exists $n$ satisfying $F_{-\infty}$,
  then every $n - \lambda\delta$, for $\lambda \in \mathbb{Z}^+$, also satisfies $F_{-\infty}$

- right disjunct $\bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[t+j]$ :

  Asserts: There is least $n \in \mathbb{Z}$ s.t. $F_4[n]$.

  For let $t^* = \{\text{largest } t \mid t < x' \text{ in (B)}\}$.
  If $n \in \mathbb{Z}$ is s.t. $F_4[n]$, then

  $$\exists j (1 \le j \le \delta).\ t^* + j \le n \ \wedge\ F_4[t^* + j]$$

  In other words,
  if there is a solution,
  then one must appear in $\delta$ interval to the right of $t^*$



## Example of Step 5 I

$$\exists x.\ \underbrace{3x + 1 > y \ \wedge\ 2x - 6 < z \ \wedge\ 4 \mid 5x + 1}_{F[x]}$$
$$\Downarrow$$
$$\exists x'.\ \underbrace{x' < 15z + 90 \ \wedge\ 10y - 10 < x' \ \wedge\ 24 \mid x' + 6 \ \wedge\ 30 \mid x'}_{F_4[x']}$$

By step 5,

$$F_{-\infty}[x'] :\ \top \ \wedge\ \bot \ \wedge\ 24 \mid x' + 6 \ \wedge\ 30 \mid x' \ ,$$

which simplifies to $\bot$.



## Example of Step 5 II

Compute

$$\delta = \text{lcm}\{24, 30\} = 120 \quad \text{and} \quad B = \{10y - 10\} \ .$$

Then replacing $x'$ by $10y - 10 + j$ in $F_4[x']$ produces

$$F_5 : \bigvee_{j=1}^{120} \left[ \begin{array}{l} 10y - 10 + j < 15z + 90 \ \wedge\ 10y - 10 < 10y - 10 + j \\ \wedge\ 24 \mid 10y - 10 + j + 6 \ \wedge\ 30 \mid 10y - 10 + j \end{array} \right]$$

which simplifies to

$$F_5 : \bigvee_{j=1}^{120} \left[ \begin{array}{l} 10y + j < 15z + 100 \ \wedge\ \cancel{0 < j} \\ \wedge\ 24 \mid 10y + j - 4 \ \wedge\ 30 \mid 10y - 10 + j \end{array} \right] .$$

$F_5$ is quantifier-free and $\widehat{T_{\mathbb{Z}}}$-equivalent to $\exists x.\ F[x]$.



## Cooper's Method: Example I

$$\exists x.\ \underbrace{(3x + 1 < 10 \ \vee\ 7x - 6 > 7) \ \wedge\ 2 \mid x}_{F[x]}$$

Isolate $x$ terms

$$\exists x.\ (3x < 9 \ \vee\ 13 < 7x) \ \wedge\ 2 \mid x \ ,$$

so

$$\delta' = \text{lcm}\{3, 7, 1\} = 21 \ .$$

After multiplying coefficients by proper constants,

$$\exists x.\ (21x < 63 \ \vee\ 39 < 21x) \ \wedge\ 42 \mid 21x \ ,$$

we replace $21x$ by $x'$:

$$\exists x'.\ \underbrace{(x' < 63 \ \vee\ 39 < x') \ \wedge\ 42 \mid x' \ \wedge\ 21 \mid x'}_{F_4[x']} \ .$$

## Cooper's Method: Example II

Then

$$F_{-\infty}[x'] : \ (\top \ \vee \ \bot) \ \wedge \ 42 \mid x' \ \wedge \ 21 \mid x' \ ,$$

or, simplifying,

$$F_{-\infty}[x'] : \ 42 \mid x' \ \wedge \ 21 \mid x' \ .$$

Finally,

$$\delta = \text{lcm}\{21, 42\} = 42 \quad \text{and} \quad B = \{39\} \ ,$$

so $F_5$ :

$$\bigvee_{j=1}^{42} (42 \mid j \ \wedge \ 21 \mid j) \ \vee$$

$$\bigvee_{j=1}^{42} ((39 + j < 63 \ \vee \ 39 < 39 + j) \ \wedge \ 42 \mid 39 + j \ \wedge \ 21 \mid 39 + j) \ .$$

Since $42 \mid 42$ and $21 \mid 42$, the left main disjunct simplifies to $\top$, so that $F_5$ is $\widehat{T_{\mathbb{Z}}}$-equivalent to $\top$. Thus, $\exists x. \ F[x]$ is $\widehat{T_{\mathbb{Z}}}$-valid.



## Cooper's Method: Example I

$$\exists x. \ \underbrace{2x = y}_{F[x]}$$

Rewriting

$$\exists x. \ \underbrace{2x < y + 1 \ \wedge \ y - 1 < 2x}_{F_3[x]}$$

Then

$$\delta' = \text{lcm}\{2, 2\} = 2 \ ,$$

so by Step 4

$$\exists x'. \ \underbrace{x' < y + 1 \ \wedge \ y - 1 < x' \ \wedge \ 2 \mid x'}_{F_4[x']}$$

$F_{-\infty}$ produces $\bot$.



## Cooper's Method: Example II

However,

$$\delta = \text{lcm}\{2\} = 2 \quad \text{and} \quad B = \{y - 1\} \ ,$$

so

$$F_5 : \ \bigvee_{j=1}^{2} (y - 1 + j < y + 1 \ \wedge \ y - 1 < y - 1 + j \ \wedge \ 2 \mid y - 1 + j)$$

Simplifying,

$$F_5 : \ \bigvee_{j=1}^{2} (j < 2 \ \wedge \ 0 < j \ \wedge \ 2 \mid y - 1 + j)$$

and then

$$F_5 : \ 2 \mid y \ ,$$

which is quantifier-free and $\widehat{T_{\mathbb{Z}}}$-equivalent to $\exists x. \ F[x]$.



## Improvement: Symmetric Elimination

In step 5, if there are fewer

(A) literals $x' < t$

than

(B) literals $t < x'$ ,

construct the right infinite projection $F_{+\infty}[x']$ from $F_4[x']$ by replacing

(A) literal $x' < t$ by $\bot$

than

(B) literal $t < x'$ by $\top$

Then right elimination.

$$F_5 : \ \bigvee_{j=1}^{\delta} F_{+\infty}[-j] \ \vee \ \bigvee_{j=1}^{\delta} \bigvee_{t \in A} F_4[t - j] \ .$$

## Improvement: Eliminating Blocks of Quantifiers I

Given

$$\exists x_1.\ \cdots \exists x_n.\ F[x_1, \ldots, x_n]$$

where $F$ quantifier-free.
Eliminating $x_n$ (left elimination) produces

$$G_1 : \quad \exists x_1.\ \cdots \exists x_{n-1}.\ \bigvee_{j=1}^{\delta} F_{-\infty}[x_1, \ldots, x_{n-1}, j] \ \vee \ \bigvee_{j=1}^{\delta} \bigvee_{t \in B} F_4[x_1, \ldots, x_{n-1}, t + j]$$

which is equivalent to

$$G_2 : \quad \bigvee_{j=1}^{\delta} \exists x_1.\ \cdots \exists x_{n-1}.\ F_{-\infty}[x_1, \ldots, x_{n-1}, j] \ \vee \ \bigvee_{j=1}^{\delta} \bigvee_{t \in B} \exists x_1.\ \cdots \exists x_{n-1}.\ F_4[x_1, \ldots, x_{n-1}, t + j]$$



## Improvement: Eliminating Blocks of Quantifiers II

Treat $j$ as a free variable and examine only $1 + |B|$ formulae

- $\exists x_1.\ \cdots \exists x_{n-1}.\ F_{-\infty}[x_1, \ldots, x_{n-1}, j]$
- $\exists x_1.\ \cdots \exists x_{n-1}.\ F_4[x_1, \ldots, x_{n-1}, t + j]$ for each $t \in B$



## Example I

$$F : \ \exists y.\ \exists x.\ x < -2 \ \wedge \ 1 - 5y < x \ \wedge \ 1 + y < 13x$$

Since $\delta' = \mathrm{lcm}\{1, 13\} = 13$

$$\exists y.\ \exists x.\ 13x < -26 \ \wedge \ 13 - 65y < 13x \ \wedge \ 1 + y < 13x$$

Then

$$\exists y.\ \exists x'.\ x' < -26 \ \wedge \ 13 - 65y < x' \ \wedge \ 1 + y < x' \ \wedge \ 13 \mid x'$$

There is one (A) literal $x' < \ldots$ and two (B) literals $\ldots < x'$, we use right elimination.

$$F_{+\infty} = \bot \qquad \delta = \{13\} = 13 \qquad A = \{-26\}$$

$$F' : \ \exists y.\ \bigvee_{j=1}^{13} \left[ \begin{array}{c} -26 - j < -26 \ \wedge \ 13 - 65y < -26 - j \\ \wedge \ 1 + y < -26 - j \ \wedge \ 13 \mid -26 - j \end{array} \right]$$



## Example II

Commute

$$G[j] : \ \bigvee_{j=1}^{13} \underbrace{\exists y.\ j > 0 \ \wedge \ 39 + j < 65y \ \wedge \ y < -27 - j \ \wedge \ 13 \mid -26 - j}_{H[j]}$$

Treating $j$ as free variable (and removing $j > 0$), apply QE to

$$H[j] : \ \exists y.\ 39 + j < 65y \ \wedge \ y < -27 - j \ \wedge \ 13 \mid -26 - j$$

Simplify...

$$H'[j] : \ \bigvee_{k=1}^{65} (k < -1794 - 66j \ \wedge \ 13 \mid -26 - j \ \wedge \ 65 \mid 39 + j + k)$$

Replace $H[j]$ with $H'[j]$ in $G[j]$

## Example III

$$F'' : \bigvee_{j=1}^{13} \bigvee_{k=1}^{65} (k < -1794 - 66j \wedge 13 \mid -26 - j \wedge 65 \mid 39 + j + k)$$

$$\uparrow \qquad\qquad \uparrow$$
$$j = 13 \qquad\qquad k = 13$$

simplified to

$$13 < -1794 - 66 \cdot 13$$

$$\perp$$

This qff formula is $\widehat{T_{\mathbb{Z}}}$-equivalent to $F$.



## Quantifier Elimination over Rationals

$$\Sigma_{\mathbb{Q}} : \{0,\ 1,\ +,\ -,\ =,\ \geq\}$$

Recall: we use $>$ instead of $\geq$, as

$$x \geq y \ \Leftrightarrow\ x > y \ \vee\ x = y \qquad x > y \ \Leftrightarrow\ x \geq y \ \wedge\ \neg(x = y)\ .$$

Ferrante & Rackoff's Method

Given a $\Sigma_{\mathbb{Q}}$-formula $\exists x.\ F[x]$, where $F[x]$ is quantifier-free, generate quantifier-free formula $F_4$ (four steps) s.t.

$$F_4 \text{ is } \Sigma_{\mathbb{Q}}\text{-equivalent to } \exists x.\ F[x]$$

by

1. putting $F[x]$ in NNF,
2. replacing negated literals,
3. solving literals such that $x$ appears isolated on one side, and
4. taking finite disjunction $\bigvee_t F[t]$.



## Ferrante & Rackoff's Method: Steps 1 and 2

Step 1: Put $F[x]$ in NNF. The result is $\exists x.\ F_1[x]$.

Step 2: Replace literals (left to right)

$$\neg(s < t) \ \Leftrightarrow\ t < s \ \vee\ t = s$$
$$\neg(s = t) \ \Leftrightarrow\ t < s \ \vee\ t > s$$

The result $\exists x.\ F_2[x]$ does not contain negations.



## Ferrante & Rackoff's Method: Step 3

Solve for $x$ in each atom of $F_2[x]$, e.g.,

$$t_1 < cx + t_2 \qquad \Rightarrow \qquad \frac{t_1 - t_2}{c} < x$$

where $c \in \mathbb{Z} - \{0\}$.

All atoms in the result $\exists x.\ F_3[x]$ have form

(A) $x < t$
(B) $t < x$
(C) $x = t$

where $t$ is a term that does not contain $x$.

## Ferrante & Rackoff's Method: Step 4 I

Construct from $F_3[x]$

- left infinite projection $F_{-\infty}$ by replacing
  - (A) atoms $x < t$ by $\top$
  - (B) atoms $t < x$ by $\bot$
  - (C) atoms $x = t$ by $\bot$
- right infinite projection $F_{+\infty}$ by replacing
  - (A) atoms $x < t$ by $\bot$
  - (B) atoms $t < x$ by $\top$
  - (C) atoms $x = t$ by $\bot$

Let $S$ be the set of $t$ terms from (A), (B), (C) atoms.
Construct the final

$$F_4 : \; F_{-\infty} \;\vee\; F_{+\infty} \;\vee\; \bigvee_{s,t \in S} F_3\left[\frac{s+t}{2}\right] \;,$$

which is $T_{\mathbb{Q}}$-equivalent to $\exists x.\ F[x]$.



## Ferrante & Rackoff's Method: Step 4 II

- $F_{-\infty}$ captures the case when small $x \in \mathbb{Q}$ satisfy $F_3[x]$
- $F_{+\infty}$ captures the case when large $x \in \mathbb{Q}$ satisfy $F_3[x]$
- last disjunct: for $s, t \in S$
  if $s \equiv t$, check whether $s \in S$ satisfies $F_3[s]$
  if $s \not\equiv t$, in any $T_{\mathbb{Q}}$-interpretation,
  - $|S| - 1$ pairs $s, t \in S$ are adjacent. For each such pair, $(s, t)$ is an interval in which no other $s' \in S$ lies.
  - Since $\frac{s+t}{2}$ represents the whole interval $(s, t)$, simply check $F_3[\frac{s+t}{2}]$ .



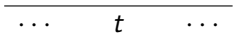
## Ferrante & Rackoff's Method: Intuition

Step 4 says that four cases are possible:

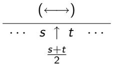
1. There is a left open interval s.t. all elements satisfy $F(x)$.

2. There is a right open interval s.t. all elements satisfy $F(x)$.

3. Some term $t$ satisfies $F(x)$.




4. There is an open interval between two $s, t$ terms such that every element satisfies $F(x)$.






## Correctness of Step 4 I

**Theorem**
*Let*

$$F_4 : \; F_{-\infty} \;\vee\; F_{+\infty} \;\vee\; \bigvee_{s,t \in S} F_3\left[\frac{s+t}{2}\right] \;,$$

*be the formula constructed from* $\exists x.\ F_3[x]$ *as in Step 4. Then* $\exists x.\ F_3[x] \;\Leftrightarrow\; F_4$.

Proof:

$\Leftarrow$ If $F_4$ is true, then $F_{-\infty}$, $F_{\infty}$ or $F_3[\frac{s+t}{2}]$ is true.
If $F_3[\frac{s+t}{2}]$ is true, then obviously $\exists x.\ F_3[x]$ is true.
If $F_{-\infty}$ is true, choose some small $x, x < t$ for all $t \in S$.
Then $F_3[x]$ is true.
If $F_{+\infty}$ is true, choose some big $x, x > t$ for all $t \in S$.
Then $F_3[x]$ is true.

## Correctness of Step 4 II

$\Rightarrow$ If $I \models \exists x.\ F_3[x]$ then there is value v such that

$$I \models F_3[v].$$

If $v < \alpha_I[t]$ for all $t \in S$, then $I \models F_{-\infty}$.
If $v > \alpha_I[t]$ for all $t \in S$, then $I \models F_{+\infty}$.
If $v = \alpha_I[t]$ for some $t \in S$, then $I \models F[\frac{t+t}{2}]$.

Otherwise choose largest $s \in S$ with $\alpha_I[s] < v$ and smallest $t \in S$ with $\alpha_I[t] > v$.

Since no atom of $F_3$ can distinguish between values in interval $(s, t)$,

$$I \models F_3[v] \quad \text{iff} \quad I \models F_3\left[\frac{s+t}{2}\right].$$

Hence, $I \models F[\frac{s+t}{2}]$. In all cases $I \models F_4$.



## Ferrante & Rackoff's Method: Example I

$\Sigma_{\mathbb{Q}}$-formula

$$\exists x.\ \underbrace{3x + 1 < 10 \ \wedge\ 7x - 6 > 7}_{F[x]}$$

Solving for $x$

$$\exists x.\ \underbrace{x < 3 \ \wedge\ x > \frac{13}{7}}_{F_3[x]}$$

Step 4: $x > \frac{13}{7}$ in (B) $\Rightarrow$ $F_{-\infty} = \bot$
$x < 3$ in (A) $\Rightarrow$ $F_{+\infty} = \bot$

$$F_4 :\ \bigvee_{s,t \in S} \underbrace{\left(\frac{s+t}{2} < 3 \ \wedge\ \frac{s+t}{2} > \frac{13}{7}\right)}_{F_3[\frac{s+t}{2}]}$$



## Ferrante & Rackoff's Method: Example II

$S = \{3, \frac{13}{7}\}$ $\Rightarrow$

$$F_3\left[\frac{3+3}{2}\right] = \bot \qquad F_3\left[\frac{\frac{13}{7}+\frac{13}{7}}{2}\right] = \bot$$

$$F_3\left[\frac{\frac{13}{7}+3}{2}\right] :\ \frac{\frac{13}{7}+3}{2} < 3 \ \wedge\ \frac{\frac{13}{7}+3}{2} > \frac{13}{7} = \top$$

$$F_4 :\ \bot \vee \cdots \vee \bot \vee \top = \top$$

Thus, $F_4 : \top$ is $T_{\mathbb{Q}}$-equivalent to $\exists x.\ F[x]$,

so $\exists x.\ F[x]$ is $T_{\mathbb{Q}}$-valid.



## Example

$$\exists x.\ \underbrace{2x > y \ \wedge\ 3x < z}_{F[x]}$$

Solving for $x$

$$\exists x.\ \underbrace{x > \frac{y}{2} \ \wedge\ x < \frac{z}{3}}_{F_3[x]}$$

*Step 4*: $F_{-\infty} = \bot$, $F_{+\infty} = \bot$, $F_3[\frac{y}{2}] = \bot$ and $F_3[\frac{z}{3}] = \bot$.

$$F_4 :\ \frac{\frac{y}{2}+\frac{z}{3}}{2} > \frac{y}{2} \ \wedge\ \frac{\frac{y}{2}+\frac{z}{3}}{2} < \frac{z}{3}$$

which simplifies to:

$$F_4 :\ 2z > 3y$$

$F_4$ is $T_{\mathbb{Q}}$-equivalent to $\exists x.\ F[x]$.