CS256/Spring 2008 — Lecture #4

Zohar Manna

Example: Proving a Congruence

For temporal formulas φ and ψ, show

$$\Diamond \Box \varphi \wedge \Diamond \Box \psi \;\approx\; \Diamond(\Box \varphi \wedge \Box \psi)$$

We have to show

$$\Diamond \Box \varphi \wedge \Diamond \Box q \Rightarrow \Diamond(\Box \varphi \wedge \Box \psi)$$

and

$$\Diamond \Box \varphi \wedge \Diamond \Box \psi \Leftarrow \Diamond(\Box \varphi \wedge \Box \psi)$$

$\Rightarrow$ The left-to-right entailment is valid:
Consider arbitrary σ and j such that

$$(\sigma, j) \vDash \Diamond \Box \varphi \wedge \Diamond \Box \psi.$$

Thus

$$\exists k_1 \geq j.\ (\sigma, k_1) \vDash \Box \varphi$$

<u>and</u>

$$\exists k_2 \geq j.\ (\sigma, k_2) \vDash \Box \psi$$

Example: Proving a Congruence (Cont'd)

$$\Diamond \Box \varphi \wedge \Diamond \Box \psi \;\approx\; \Diamond(\Box \varphi \wedge \Box \psi)$$

Unraveling the definition of $\Box$, we get

$$\exists k_1 \geq j.\ \forall k_1' \geq k_1.\ (\sigma, k_1') \vDash \varphi$$

<u>and</u>

$$\exists k_2 \geq j.\ \forall k_2' \geq k_2.\ (\sigma, k_2') \vDash \psi.$$

This implies that

$$\overbrace{\exists k \geq j.}^{k=max\{k_1,k_2\}} \quad \forall k' \geq k.\ (\sigma, k') \vDash \varphi \ \underline{\text{and}}\ (\sigma, k') \vDash \psi.$$

So

$$\exists k \geq j.\ (\sigma, k) \vDash (\Box \varphi \wedge \Box \psi).$$

That is,

$$(\sigma, j) \vDash \Diamond(\Box \varphi \wedge \Box \psi).$$

$\Leftarrow$ The right-to-left entailment is valid.
All implications in the first part hold in reverse, so the entailment is valid.

Example: Proving an Equivalence / Disproving a Congruence

For temporal logic formulas φ and ψ, show

$$\Diamond \varphi \;\sim\; \Diamond \diamondminus \varphi \qquad \Diamond \varphi \;\not\approx\; \Diamond \diamondminus \varphi$$

We shall prove: (1) $\Diamond \varphi \Rightarrow \Diamond \diamondminus \varphi$ is valid;
Thus $\Diamond \varphi \rightarrow \Diamond \diamondminus \varphi$ is valid.
(2) $\Diamond \diamondminus \varphi \rightarrow \Diamond \varphi$ is valid.
(3) $\Diamond \diamondminus \varphi \Rightarrow \Diamond \varphi$ is <u>not</u> valid.

(1) $\Diamond\varphi \Rightarrow \Diamond \diamondminus \varphi$ is valid:

Consider arbitrary σ and j such that

$$(\sigma, j) \vDash \Diamond \varphi.$$

Then $\quad \exists i \geq j.\ (\sigma, i) \vDash \varphi.$

Hence $\quad \exists i \geq j.\ \underbrace{\exists k\colon 0 \leq k \leq i.}_{k=i}\ (\sigma, k) \vDash \varphi.$

By def. $\quad \exists i \geq j.\ (\sigma, i) \vDash \diamondminus \varphi.$

Therefore $\quad (\sigma, j) \vDash \Diamond \diamondminus \varphi.$

(2) $\Diamond \diamondminus \varphi \rightarrow \Diamond \varphi$ is valid:

Consider arbitrary σ such that

$$(\sigma, 0) \vDash \Diamond \diamondminus \varphi.$$

Then $\quad \exists i \geq 0.\ (\sigma, i) \vDash \diamondminus \varphi.$

Hence $\quad \exists i \geq 0.\ \exists k\colon 0 \leq k \leq i.\ (\sigma, k) \vDash \varphi.$

Hence ($k = i$) $\quad \exists k \geq 0.\ (\sigma, k) \vDash \varphi.$

Therefore $\quad (\sigma, 0) \vDash \Diamond \varphi.$

(3) $\Diamond \diamondminus \varphi \Rightarrow \Diamond \varphi$ is **not** valid. Counterexample:

> Take $\varphi\colon p$ (propositional symbol)
> $\sigma = \langle\ s_0\colon p,\ \ s_1\colon \neg p,\ \ s_2\colon \neg p,\ \ s_3\colon \neg p,\ \ \ldots\ \rangle$
> and $j = 1$

Then $\quad (\sigma, 1) \vDash \Diamond \diamondminus p,$

but $\quad (\sigma, 1) \nvDash \Diamond p.$

Rigid and Flexible Variables

Variables in the vocabulary are partitioned into:

Rigid Variables:
Rigid variable has the same value in all states of a sequence σ

Flexible Variables:
The values of a flexible variable may be different in different states of a sequence σ.

- system variables are generally flexible (except for variables declared as *in* in an SPL program)
- auxiliary variables (used in specification) are usually rigid

`Example:`

"every value placed in x is eventually copied to z"

$\forall u.\ (x = u\ \Rightarrow\ \Diamond(z = u))$

u is a rigid auxiliary variable

Temporal Logic: Quantification

Definition:

Model $\sigma' :\ s'_0,\ s'_1,\ s'_2,\ \ldots$ is a u-variant of

$$\sigma :\ s_0,\ s_1,\ s_2,\ \ldots$$

if for every $j \geq 0$

s'_j agrees with s_j on the interpretation of all variables $y \in V - \{u\}$

> `Example:`
>
> σ': $\langle x\colon 0, y\colon 1, \boxed{z\colon 0}\rangle,\ \langle x\colon 1, y\colon 2, \boxed{z\colon 1}\rangle,\ \langle x\colon 2, y\colon 3, \boxed{z\colon 4}\rangle,\ \ldots$
>
> is a z-variant of
>
> σ: $\langle x\colon 0, y\colon 1, \boxed{z\colon 0}\rangle,\ \langle x\colon 1, y\colon 2, \boxed{z\colon 0}\rangle,\ \langle x\colon 2, y\colon 3, \boxed{z\colon 0}\rangle,\ \ldots$

Semantics of Quantification

For temporal formula φ:

- $(\sigma, j) \vDash \exists u.\varphi \iff (\sigma', j) \vDash \varphi$ for some σ', a u-variant of σ
- $(\sigma, j) \vDash \forall u.\varphi \iff (\sigma', j) \vDash \varphi$ for all σ', a u-variant of σ

Examples

Let x, y be flexible variables

> $$\sigma \vDash \exists y.\,\Box(y = x^2)$$
>
> for σ: $\langle x\colon 1,\ y\colon 2\rangle,\ \langle x\colon 2,\ y\colon 3\rangle,\ \langle x\colon 3,\ y\colon 4\rangle,\ \ldots$

Take a y-variant

σ' : $\langle x : 1, \boxed{y : 1}\rangle, \langle x : 2, \boxed{y : 4}\rangle, \langle x : 3, \boxed{y : 9}\rangle, \ldots$

We have $(\sigma', 0) \vDash \Box(y = x^2)$

Therefore, $(\sigma, 0) \vDash \exists y.\,\Box(y = x^2)$

Examples

Let x be a flexible variable
u be a rigid variable

$$\boxed{\sigma \not\vDash \exists u.\ \Box(u = x^2)}$$

Consider σ: $\langle x\colon 1,\ u\colon 0\rangle,\ \langle x\colon 2,\ u\colon 0\rangle,\ \langle x\colon 3,\ u\colon 0\rangle,\ \ldots$

Since u is rigid, every u-variant must be of the form

σ': $\langle x\colon 1, \boxed{u\colon a}\rangle,\ \langle x\colon 2, \boxed{u\colon a}\rangle,\ \langle x\colon 3, \boxed{u\colon a}\rangle,\ \ldots$

(with u having the same value in all states)

There is no u-variant σ' such that

$$(\sigma', 0) \vDash \Box(u = x^2)$$

Therefore, $(\sigma, 0) \not\vDash \exists u.\ \Box(u = x^2)$

Examples

Let u be a rigid variable.

$$\Diamond(\forall u.\ p) \not\approx \forall u.\ \Diamond p$$

i.e., $\Diamond(\forall u.\ p) \leftrightarrow \forall u.\ \Diamond p$ is not valid

Take $p : x \neq u$ with x flexible

$$\sigma\colon\ \langle x\colon 0,\ u\colon 2\rangle,\ \langle x\colon 1,\ u\colon 2\rangle,\ \langle x\colon 2,\ u\colon 2\rangle,\ \ldots$$

- left side: $\Diamond(\forall u.(x \neq u))$

  There is no position j such that
  $(\sigma, j) \vDash \forall u\ .\ x \neq u$ (take $u = x$)

  Therefore $(\sigma, 0) \nvDash \Diamond(\forall u.(x \neq u))$

  i.e., $\boxed{\sigma \nvDash \Diamond(\forall u.(x \neq u))}$

- right side: $\forall u.\ \Diamond(x \neq u)$

Take an arbitrary u-variant of σ:

$$\sigma'_a\colon\ \langle x\colon 0,\ \boxed{u\colon a}\rangle,\ \langle x\colon 1,\ \boxed{u\colon a}\rangle,\ \langle x\colon 2,\ \boxed{u\colon a}\rangle, \ldots$$

and consider two cases

case $a = 0$	case $a \neq 0$
$\langle \sigma'_a, 1\rangle \vDash x \neq u$	$\langle \sigma'_a, 0\rangle \vDash x \neq u$
$\Downarrow$	$\Downarrow$
$\langle \sigma'_a, 0\rangle \vDash \Diamond(x \neq u)$	$\langle \sigma'_a, 0\rangle \vDash \Diamond(x \neq u)$

$$\langle \sigma, 0\rangle \vDash \forall u.\ \Diamond(x \neq u)$$

i.e., $\boxed{\sigma \vDash \forall u.\ \Diamond(x \neq u)}$

Therefore,

$$\Diamond(\forall u.p) \leftrightarrow \forall u.\ \Diamond(x \neq u)$$

is not valid.

Conjunction and Disjunction Characters

- For first-order logic:
  $\wedge$ has conjunction character

  $\vee$ has disjunction character

- For Temporal Logic:
  $\Box$ and $\boxminus$ have conjunction character

  $\Diamond$ and $\Diamond\!\!\!\!-$ have disjunction character

  $\mathcal{U}, \mathcal{W}, \mathcal{S}, \mathcal{B}$ have conjunction character w.r.t. the first argument and disjunction character w.r.t. the second argument

- For Quantifiers:
  $\forall$ has conjunction character

  $\exists$ has disjunction character

Congruences

$$\forall u.(\varphi \wedge \psi) \approx \forall u.\varphi \wedge \forall u.\psi$$

$$\exists u.(\varphi \vee \psi) \approx \exists u.\varphi \vee \exists u.\psi$$

$$\Box(\forall u.\varphi) \approx \forall u.\ \Box\varphi$$

$$\Diamond(\exists u.\varphi) \approx \exists u.\ \Diamond\varphi$$

$$\varphi\, \mathcal{U}\, (\exists u.\psi) \approx \exists u.(\varphi\, \mathcal{U}\, \psi) \quad (u \text{ not free in } \varphi)$$

$$(\forall u.\varphi)\, \mathcal{U}\, \psi \approx \forall u.(\varphi\, \mathcal{U}\, \psi) \quad (u \text{ not free in } \psi)$$

$$\varphi\, \mathcal{W}\, (\exists u.\psi) \approx \exists u.(\varphi\, \mathcal{W}\, \psi) \quad (u \text{ not free in } \varphi)$$

$$(\forall u.\varphi)\, \mathcal{W}\, \psi \approx \forall u.(\varphi\, \mathcal{W}\, \psi) \quad (u \text{ not free in } \psi)$$

Expressibility

There are properties that cannot be specified by a quantifier-free temporal logic formula.

Example:

Specify the property
"x assumes the value 0 only, if ever, at even positions"
i.e., "at positions $0, 2, 4, \ldots$"

- cannot be expressed in quantifier-free TL
- can be expressed in (quantified) TL

Quantifying over flexible boolean variable b:
$\exists b[b \;\wedge\; \Box(b \leftrightarrow \neg \bigcirc b) \;\wedge\; \Box(x = 0 \rightarrow b)]$.
$\forall b[b \;\wedge\; \Box(b \leftrightarrow \neg \bigcirc b) \;\rightarrow \Box(x = 0 \rightarrow b)]$.

Why not

$$x = 0 \wedge \Box[x = 0 \rightarrow \bigcirc \bigcirc (x = 0)]?$$

Temporal vs First-Order

TL formula

$$\Box(p \;\rightarrow\; \Diamond[r \;\wedge\; \Diamond q])$$

can be transformed into FOL formula

$$(\forall t_1 \geq 0)\left[\; p(t_1) \rightarrow (\exists t_2)\left[\begin{array}{c} t_1 \leq t_2 \;\wedge\; r(t_2) \;\wedge \\ (\exists t_3)(t_2 \leq t_3 \;\wedge\; q(t_3)) \end{array}\right]\;\right]$$

where t_1, t_2, t_3 are integers.

Temporal Logic + Programs

P-Validity

Given a program P:

- For a state formula q:

$\Vdash q$ (q is state valid)
if q holds in all states

Example:
$\Vdash \; x = 1 \;\rightarrow\; x > 0$

$P \Vdash q$ (q is state valid over P
q is P-state valid)
if q holds over all P-accessible states

Recall : State s_i is a P-accessible state if it is in some computation $\sigma: \; s_0, \; s_1, \; s_2, \; \ldots, \; s_i, \; \ldots$ of P.

`Example`

local x: **integer where** $x = 1$

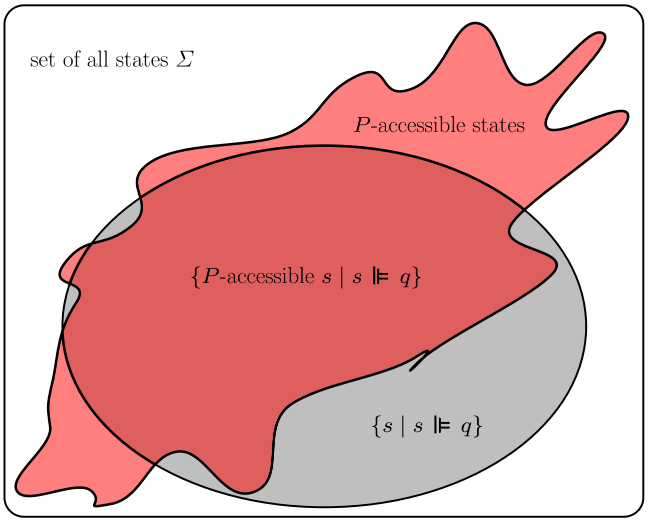
$$\left[\begin{array}{l} \ell_0\text{: } \textbf{loop forever do} \\ \quad \left[\begin{array}{l} \ell_1\text{: } \textbf{await } x = 1 \\ \ell_2\text{: } x := 2 \end{array}\right] \end{array}\right] \;||\; \left[\begin{array}{l} m_0\text{: } \textbf{loop forever do} \\ \quad \left[\begin{array}{l} m_1\text{: } \textbf{await } x = 2 \\ m_2\text{: } x := 1 \end{array}\right] \end{array}\right]$$

$P \;\Vdash\; x = 1 \vee x = 2$
$P \;\Vdash\; at\_\ell_2 \;\rightarrow\; x = 1$

Recall: at_ℓ_2 stands for $[\ell_2] \in \pi$

P-Validity (Con't)

$P \Vdash q$



P-Validity (Con't)

Given a program P:

- For a temporal formula φ:

  $\vDash \varphi$ (φ is valid)
  if φ holds in the first state of every model (i.e., every infinite sequence of states)

Example:

$\vDash \Box p \lor \Diamond \neg p$

P-Validity (Con't)

$P \vDash \varphi$ (φ is valid over P, φ is P-valid)
if φ holds in the first state of every P-computations

Example:

local x: **integer where** $x = 1$

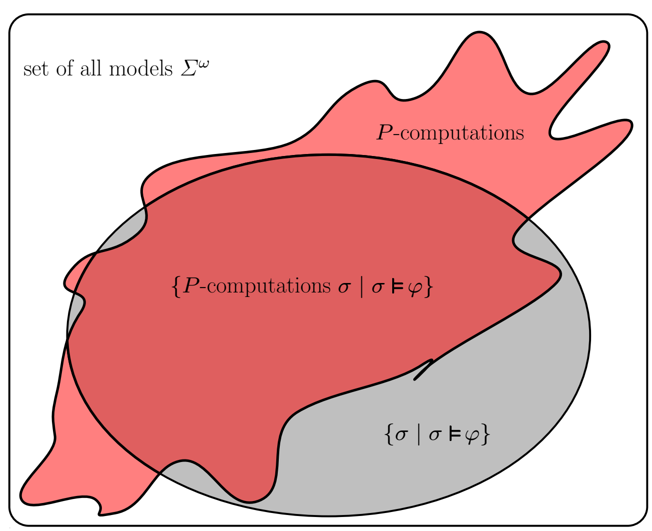
$$\left[\begin{array}{l}\ell_0\text{: \textbf{loop forever do}}\\ \quad\left[\begin{array}{l}\ell_1\text{: \textbf{await} } x = 1\\ \ell_2\text{: } x := 2\end{array}\right]\end{array}\right] \;\|\; \left[\begin{array}{l}m_0\text{: \textbf{loop forever do}}\\ \quad\left[\begin{array}{l}m_1\text{: \textbf{await} } x = 2\\ m_2\text{: } x := 1\end{array}\right]\end{array}\right]$$

$P \vDash \Box\Diamond(x = 1) \land \Box\Diamond(x = 2)$

$P \vDash at\_\ell_1 \Rightarrow \Diamond\, at\_\ell_2$

P-Validity (Con't)

$P \vDash \varphi$



P-Validity (Con't)

	general	program P
state formula q	$\Vdash q$ **state valid** "q holds in all states" $x = 1 \rightarrow x > 0$	$P \Vdash q$ **P-state valid** "q holds in all P-accessible states" $x = 1 \vee x = 2$
temporal formula φ	$\vDash \varphi$ **valid** "φ holds in first position of every sequence" $\Box p \vee \Diamond \neg p$	$P \vDash \varphi$ **P-valid** "φ holds in first position of every P-computation" $at\_\ell_1 \Rightarrow \Diamond at\_\ell_2$

Similarly,
P-satisfiability, P-equivalence,
P-congruence

P-Validity (Con't)

For state formula q:

$$\Vdash q \quad \longleftrightarrow \quad \vDash \Box q$$
$$P \Vdash q \quad \longleftrightarrow \quad P \vDash \Box q$$
$$\Vdash q \quad \longrightarrow \quad P \Vdash q \quad \text{but not vice-versa}$$

For temporal formula φ:

$$\vDash \varphi \quad \longrightarrow \quad P \vDash \varphi \quad \text{but not vice-versa}$$

Specification of Properties

- property Π of P = set of models
- Π is specified by temporal formula p if for every model σ, $\sigma \in \Pi$ iff $\sigma \vDash p$
- P has property Π if

  $\{P\text{-computations}\} \subseteq \{\Pi\text{-models}\}$

Classification of TL formulas

Reason for classification:

each class is associated with a proof principle for verifying that a given program satisfies a property specifiable by a formula in the class.

Broad classification: Safety – Progress

Safety

- all finite prefixes of a computation satisfy a certain requirement.
- "no bad things will happen"
- violation can be detected in finite time
- satisfaction of a safety property does not depend on the fairness conditions:
  a safety formula φ holds on all P-computations iff φ holds on all P-runs,
  i.e., a safety property cannot distinguish P-computations and P-runs.
- topic of textbook

Progress (liveness)

- "something good will happen eventually"
- always depends on fairness conditions in non-trivial cases, because the set of P-runs includes the sequence

$$s_0 \xrightarrow{\tau_I} s_1 \xrightarrow{\tau_I} s_2 \xrightarrow{\tau_I} \dots$$

  i.e., the idling transition is the only transition ever taken