Problem Set 8

Electronic submission (email as attachment to psetCS265@gmail.com) due 11am Thursday 12/4. If submitting a hard copy, there is a box on the 1st floor of Gates Building, by the East entrance, labelled CS265/CME309. Hard copies must be submitted by 10am Thursday 12/4.

[You may discuss these problems with classmates. Please do not troll the internet looking for solutions to these problems. All writing must be done independently, and you must fully understand everything you write.]

1. Consider a random walk on the $n$-dimensional hypercube (i.e. where the vertices are $n$-bit numbers, and edges connect numbers that differ in a single index), where in each step, with probability $1/2$ one stays at the current vertex, and with probability $1/2$, one chooses a random index $i \leftarrow \{1, \ldots, n\}$ and flips the $i$th coordinate. Prove that the mixing time is $O(n \log n)$.

[Hint: When defining the coupling, it might be helpful to view the above random walk as follows: pick a random index $i \leftarrow \{1, \ldots, n\}$, and a value $b \leftarrow \{0, 1\}$, and update the $i$th coordinate to have value $b$.]

2. Suppose the class has $c$ CS students, and $s$ Statistics students, and that a uniformly chosen random subset of $n \leq c + s$ students attend office hours. Let $X$ denote the number of CS students who attend office hours. The goal of this problem is to prove that $X$ will be tightly concentrated about its expectation: $\Pr[|X - E[X]| > \lambda] \leq 2e^{-\frac{\lambda^2}{2n}}$.

(a) Define a martingale that captures the quantity we care about analyzing. [Hint: in general, if you want to make a martingale argument, but don’t know where to begin, try defining the Doob martingale.]

(b) Prove that $\Pr[|X - E[X]| > \lambda] \leq 2e^{-\frac{\lambda^2}{2n}}$. [Hint: Azuma-Hoeffding tail bounds!]

3. Recall the “martingale stopping theorem”, which states that for a martingale $\{Z_i\}$ with respect to $\{X_i\}$, then if $T$ is a stopping time for $\{X_i\}$, then $E[Z_T] = E[Z_0]$ provided at least one of the following conditions hold:

- There exists a constant $c$ s.t. $|Z_i| \leq c$ for all $i$.
- There exists a constant $c$ s.t. $T < c$.
- $E[T] < \infty$, and there exists a constant $c$ s.t. $E[|Z_{i+1} - Z_i| \mid X_1, \ldots, X_i] < c$.

(a) Give an example of a martingale and stopping time for which $E[Z_T] \neq E[Z_0]$ but for which $E[T] < \infty$. Briefly justify why your example satisfies the desired conditions.

(b) Give an example of a martingale and stopping time for which $E[Z_T] \neq E[Z_0]$ but for which $E[|Z_{i+1} - Z_i| \mid X_1, \ldots, X_i] < 1$. Briefly justify why your example satisfies the desired conditions.
4. Suppose a casino has a gambling game where you either win or lose one dollar, with respective probabilities $p$ and $1 - p$ (that do not change between rounds). Assume you start with $s$ dollars, and play until you either run out of money, or double your money. In class we saw that if $p = 1/2$, the expected number of rounds you will play is exactly $s^2$. In this problem, we see how to compute this in the case $p \neq 1/2$.

(a) If $X_i$ is the amount you win in the $i$th round, find the value of $\alpha$ as a function of $p$ such that $Y_i = \sum_{j=1}^i X_j + \alpha i$ is a martingale.

(b) What is the expected number of rounds played? [Hint: Define another martingale...try the exponential function $c \sum_{j=1}^i X_j$ for a clever choice of $c$.]

(c) Using the expression in the previous part, and matlab, mathematica, wolfram alpha, etc, compute a numerical estimate of the expected number of rounds played in the following scenarios:
   i. $s = 1000, p = 0.5001$
   ii. $s = 1000, p = 0.501$
   iii. $s = 1000, p = 0.51$
   iv. $s = 1000, p = 0.6$

5. Given a social network defined by the undirected graph $G = (V, E)$, assume that every person (vertex) is either healthy, or addicted to snap-chat. Furthermore, assume that people’s states evolve according to the following process: every day (simultaneously for all people) each person flips (their own) coin, and with probability $1/2$ they keep their current state, and with probability $1/2$, they adopt the state of a randomly chosen neighbor.

(a) Let $X_t$ be the number of people addicted to snap-chat at time $t$. Is $\{X_t\}$ a martingale?

(b) Let $Y_t$ be the sum of the degrees of all the people addicted to snapchat at time $t$. Is $\{Y_t\}$ a martingale?

(c) Using the martingale stopping theorem, what is the probability that the entire set of people end up addicted to snap-chat, as a function of the starting configuration?

(d) (BONUS) Prove that the expected time until everyone is either addicted, or healthy, is bounded by $O(|E|^2)$. [Hint: A sequence $\{Y_t\}$ is a submartingale with respect to $\{X_t\}$ if $E[Y_{t+1}|X_0, \ldots, X_t] \geq Y_t$. [If the inequality is flipped, the sequence is known as a supermartingale.] The Martingale stopping theorem holds for submartingales, with the modification that the equality becomes an inequality, e.g. $E[Y_T] \geq E[Y_0]$, provided $T$ is a stopping time satisfying the conditions of the Martingale Stopping Theorem. This inequality flips in the case of super-martingales.]

6. Have a happy thanksgiving.