Problem Set 1

Electronic submission (email as attachment to psetCS265@gmail.com) due 10am Thursday 10/1. You are strongly encouraged to submit a homework with a partner—that is, submit one homework with both of your names.

[You may discuss these problems with classmates. Please do not troll the internet looking for solutions to these problems. You must understand everything that is written on the homework you submit. We suggest that you write solutions using LaTeX.]

1. For the following questions, either show your work, or give at most one sentence of explanation for each part.
   
   (a) Suppose we flip 10 independent fair coins (Pr[heads] = Pr[tails] = \frac{1}{2}), what is the probability that the total number of heads observed is 5?
   
   (b) Suppose we flip 10 independent fair coins, what is the probability that there are are more heads than tails?
   
   (c) Suppose we flip n independent fair coins, for some even integer n ≥ 2, what is the probability that we see exactly n/2 heads? (Feel free to leave your answer in terms of the binomial coefficients).
   
   (d) Suppose we flip n independent fair coins, for some even integer n ≥ 2; using the fact that
   
   k! \approx \frac{k^k}{e^k} \sqrt{2\pi k} \quad (known \ as \ “Stirling’s \ approximation”), roughly what is the probability of seeing exactly n/2 heads, as a function of n? Please write this expression without any binomial coefficients.

2. Random Bits:

   (a) Given access to coins whose probability of landing heads is 1/3, describe a scheme for generating a random bit. In other words, how can one simulate a flip of a coin that lands heads with probability 1/2?
   
   (b) Given access to coins whose probability of landing heads is 1/2, describe a scheme for simulating the flip of a coin whose probability of heads is 1/3.
   
   (c) In one or two sentences, explain why the above problems are impossible if one is required to write a protocol that flips the given coins at most some finite number, k, times.

3. Suppose there are 100 pigeons, and 100 pigeon-holes, with each pigeon-hole belonging to a pigeon. One crazy night, all the pigeons get rather drunk, and each one randomly selects a pigeon-hole to return to.

   (a) What is the expected number of pigeons that end up returning to their proper hole? (Hint: use linearity of expectation!)
   
   (b) What is the expected number of pigeons that will end up in a hole with exactly one other pigeon? (Hint: use linearity of expectation!)
(c) Suppose we have \( n \) pigeons, and \( n \) holes. Show that in the limit as \( n \to \infty \), the expected fraction of empty holes after the drunken night approaches \( 1/e \).

4. Improving the random min-cut algorithm seen in class:

(a) Consider running the algorithm seen in class twice (recall that the algorithm proceeds by repeatedly choosing a random edge to “contract” until there are only two vertices remaining). What is the number of edge contractions? And what is a lower bound on the probability of finding a min-cut (based on the analysis seen in class)?

(b) Consider the following variation: starting with a graph on \( n \) vertices, first contract the graph down to \( k \) vertices, and now run the randomized algorithm on this smaller graph \( \ell \) (independent) times and output the smallest cut found in any of the \( \ell \) runs. Determine the number of edge contractions and bound the probability of finding a minimum cut, as a function of \( n, k, \ell \).

(c) What are near-optimal values of \( k, \ell \), (as functions of \( n \)) which maximize the probability of finding a minimum cut, while using the same number of edge contractions as running the original algorithm twice? [Feel free to just give the asymptotics—i.e. “\( k = \Theta(n^{\text{blah}}) \), in which case the probability of success is \( \Theta(n^{-\text{blah}}') \).”]

(d) Bonus. Note that one could extend the above improvements by having a multi-stage algorithm with \( k_1, k_2, \ldots \), and corresponding \( \ell_1, \ell_2, \ldots \) (as opposed to the two-stage algorithm considered above). If such a scheme is carried out optimally, how many contractions (as a function of \( n \)) are required to achieve probability of success \( > 1/2 \)?