Instructions: Everyone needs to submit their own write-up. If you work together with other students, indicate their names on your write-up.

All graphs in this problem set are undirected, simple and unweighted.

Problem 1

Let $P_3$ be the path on 3 nodes. We say that an undirected graph $G = (V, E)$ contains an induced $P_3$ if there exist $u, v, w \in V$ such that $(u, v), (v, w) \in E$ but $(u, w) \notin E$.

Design an $O(m+n)$ time algorithm that finds an induced $P_3$ in any $m$-edge, $n$-node graph $G$, or determines that $G$ does not contain an induced $P_3$.

Hint: A graph contains a $P_3$ if and only if it is not a node-disjoint union of cliques (of possibly different sizes; a single node is considered a clique of size 1; you may want to prove this).

Problem 2

In class we showed that a 4-cycle (if one exists) in an $n$-node graph can be found in $O(n^2)$ time. Give a similar algorithm that can find a 4-cycle in an $m$-edge $n$-node graph in $O(m\sqrt{n})$ time. (Hint: use the high degree-low degree technique)

**BONUS:** Give an algorithm for finding a 4-cycle running in $O(m^{4/3})$ time.

Problem 3

1. Show that exact girth of an $n$-node, $m$-edge graph can be found in $O(nm)$ time (Hint: your algorithm may look similar to Itai and Rodeh’s algorithm). Explain the worst-case behavior that would cause your algorithm to require $O(nm)$ time, rather than $O(n^2)$.

2. Let $G$ be an $n$-node graph such that for some integer $g \geq 1$, the minimum degree of $G$ is $> \lceil n^{1/g} \rceil$. Prove that $G$ has girth at most $2g$.

3. Let $k \geq 1$ be a given integer and suppose $G$ is an $n$-node regular graph (i.e. the degrees of all vertices in $G$ are the same). Show that one can either find in $O(m + n)$ time a cycle of length at most $2k$, or in $O(n^{2+1/k})$ time one can compute the exact girth. (Hint: use (a) and (b).)