CS 369P: Polyhedral techniques in combinatorial optimization
Instructor: Jan Vondrák
Lecture date: September 28, 2010 **Scribe: Richard Pang**

1 Properties of polyhedral vertices

First, let us give four equivalent properties each of which can be taken as a definition of a vertex.

Lemma 1 *Let $P \subset \mathbb{R}^n$ be a polyhedron and let $\mathbf{v} \in P$. The following properties are equivalent:*

(1) There is a hyperplane $H = \{\mathbf{x} : \mathbf{w}^T\mathbf{x} = \lambda\}$ such that $\mathbf{w}^T\mathbf{x} \leq \lambda$ for all $\mathbf{x} \in P$ and $P \cap H = \{\mathbf{v}\}$.

(2) v cannot be expressed as a convex combination of points in $P \setminus \{v\}$.

(3) There is no nonzero vector $\mathbf{d}$ such that $\mathbf{v} \pm \mathbf{d} \in P$.

(4) There are n constraints $\mathbf{a}_j^T\mathbf{x} \leq b_j$ valid for P, which are tight at $\mathbf{v}$, i.e. $\mathbf{a}_j^T\mathbf{v} = b_j$ for $1 \leq j \leq n$, and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent.

Proof:

(1) ⇒ (2) Suppose $H = \{\mathbf{x} : \mathbf{w}^T\mathbf{x} = \lambda\}$ is a hyperplane such that $P \cap H = \{\mathbf{v}\}$. If $\mathbf{v}$ is a convex combination of other points in P, $\mathbf{v} = \sum \alpha_i \mathbf{v}_i$ where $\sum \alpha_i = 1$ and $\alpha_i \geq 0$, then we have

$$\mathbf{w}^T\mathbf{v} = \sum \alpha_i(\mathbf{w}^T\mathbf{v}_i) = \lambda.$$

However, $\mathbf{w}^T\mathbf{v}_i \leq \lambda$ for each i. The only way the equality above can be achieved is that $\mathbf{w}^T\mathbf{v}_i = \lambda$ for each i. This means that $\mathbf{v}$ is not the only point in $H \cap P$.

(2) ⇒ (3) If $\mathbf{v} \pm \mathbf{d} \in P$, then $\mathbf{v} = \frac{1}{2}(\mathbf{v} + \mathbf{d}) + \frac{1}{2}(\mathbf{v} - \mathbf{d})$ is a convex combination of other points in P, a contradiction.

(3) ⇒ (4) We have $\mathbf{v} \in P = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$. Consider $T = \{\mathbf{a}_j : \mathbf{a}_j^T\mathbf{v} = b_j\}$, the normal vectors to all constraints which are tight at $\mathbf{v}$. If $\dim(\mathrm{span}(T)) < n$, then there exists $\mathbf{d} \neq \mathbf{0}$ such that $\mathbf{d}$ is orthogonal to span(T). I.e. for all $\mathbf{a}_j \in T, \mathbf{a}_j^T\mathbf{d} = 0$ and $\mathbf{a}_j^T(\mathbf{v} \pm \epsilon\mathbf{d}) = \mathbf{a}_j^T\mathbf{v} = b_j$. For all other constraints, we have $\mathbf{a}_i^T\mathbf{v} < b_i$, so there is some small $\epsilon > 0$ such that $\mathbf{a}_i^T(\mathbf{v} \pm \epsilon\mathbf{d}) \leq b_i$ for all i. This means that $\mathbf{v} \pm \epsilon\mathbf{d} \in P$.

(4) ⇒ (1) Assume that $\mathbf{a}_j^T\mathbf{x} \leq b_j, 1 \leq j \leq n$ are valid constraints for P, the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent, and $\mathbf{a}_j^T\mathbf{v} = b_j$ for each j. Consider the hyperplane

$$H = \{\mathbf{x} : \sum_{j=1}^{n} \mathbf{a}_j^T\mathbf{x} = \sum_{j=1}^{n} b_j\}.$$

For any $\mathbf{x} \in P$, since $\mathbf{a}_j^T\mathbf{x} \leq b_j$ for each j, we have $\sum_{j=1}^n \mathbf{a}_j^T\mathbf{x} \leq \sum_{j=1}^n b_j$. Furthermore, the only way we can get an equality is that $\mathbf{a}_j^T\mathbf{x} = b_j$ for each j. However, this is a rank n system of linear equations in dimension n, which has a unique solution - the point $\mathbf{v}$.

□

This implies the following property of polyhedra defined by rational data.

Corollary 2 *Let $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron where $\mathbf{A}$ and $\mathbf{b}$ are rational. Then every vertex of P is also rational.*

Proof: Let $\mathbf{v}$ be a vertex of P. By Lemma 1, $\mathbf{v}$ is the solution of a full-rank system of linear equations $\mathbf{A}'\mathbf{v} = \mathbf{b}'$. By Cramer's rule, the coordinates of $\mathbf{v}$ can be written as ratios of determinants with rational entries, i.e. they are rational. □

2 Totally Unimodular Matrices

Definition 3 (Totally Unimodular Matrix) *A matrix $\mathbf{A}$ is totally unimodular if every square submatrix has determinant 0, $+1$, or -1. In particular, this implies that all entires are 0 or ± 1.*

Theorem 4 *If $\mathbf{A}$ is totally unimodular and $\mathbf{b}$ is an integer vector, then $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ has integer vertices.*

Proof: Let $\mathbf{v}$ be a vertex of P. By Lemma 1, there exists a square submatrix $\mathbf{A}'$ of $\mathbf{A}$ such that $\mathbf{A}'\mathbf{v} = \mathbf{b}'$. We have $\det \mathbf{A}' = \pm 1$ since $\mathbf{A}'$ is nonsingular. By Cramer's Rule, we have $v_i = \frac{\det(\mathbf{A}'_i|\mathbf{b})}{\det \mathbf{A}'}$ where $\mathbf{A}'_i|\mathbf{b}$ is $\mathbf{A}'$ with the i-th column replaced by $\mathbf{b}$. Therefore, v_i is an integer. □

Lemma 5 *For all bipartite graphs G, the incidence matrix $\mathbf{A}$ is totally unimodular.*

Proof: Recall that $\mathbf{A}$ is a 0-1 matrix, where columns are indexed by edges and each column has exactly two 1's, corresponding to the two vertices of the edge. We proceed by induction. The claim is certainly true for a 1×1 matrix.

Assume the claim holds true for all $(k-1) \times (k-1)$ submatrices. Let $\mathbf{A}'$ be a $k \times k$ submatrix of $\mathbf{A}$. Each column in $\mathbf{A}'$ has at most two 1's. If any column has no 1's, it must have all 0's, and the matrix is singular. If any column has exactly one nonzero entry, then $\det \mathbf{A}' = \pm \det \mathbf{A}''$, where $\mathbf{A}''$ is obtained by deleting the respective row and column; we have $\det \mathbf{A}'' \in \{0, \pm 1\}$ by induction.

$$\begin{array}{c|c|c} \ddots & 0 & \iddots \\ \hline \cdots & 1 & \cdots \\ \hline \iddots & 0 & \ddots \end{array}$$

Otherwise, every column has exactly two 1's. In particular, since G is bipartite, the rows can be partitioned into V_1, V_2 such that for each column, there is exactly one 1 in V_1 and in V_2. Then by summing up all the rows corresponding to V_1 and subtracting the rows corresponding to V_2, we get $\mathbf{0}$. Therefore, $\mathbf{A}'$ is singular and $\det \mathbf{A}' = 0$.

$$\begin{array}{l} V_1 \left\{ \begin{array}{c|c|c} \ddots & \vdots & \iddots \\ \cdots & 1 & \cdots \\ \iddots & \vdots & \ddots \end{array} \right. \begin{array}{c} + \\ + \\ + \end{array} \\ \hline V_2 \left\{ \begin{array}{c|c|c} \ddots & \vdots & \iddots \\ \cdots & 1 & \cdots \\ \iddots & \vdots & \ddots \end{array} \right. \begin{array}{c} - \\ - \\ - \end{array} \end{array}$$

$\square$

Lemma 6 *If* $\mathbf{A}$ *is totally unimodular, then* $\left[\begin{array}{c}\mathbf{A} \\ \hline \mathbf{I}\end{array}\right]$ *is totally unimodular.*

Proof: By the determinant expansion formula, the determinant of any square submatrix $\mathbf{A}'$ is equal to 0 or $\pm \det \mathbf{A}''$ where $\mathbf{A}''$ is a square submatrix of $\mathbf{A}$ (see the figure). By definition, $\det \mathbf{A}'' \in \{0, \pm 1\}$.

$$\left[\begin{array}{ccc|c} & \cdots & & \mathbf{A}'' \\ \hline 1 & & & \\ & \ddots & & \vdots \\ & & 1 & \end{array}\right]$$

$\square$

3 Consequences for bipartite matchings

Definition 7 (b-matching) *A* $\mathbf{b}$*-matching is an assignment* $\mathbf{x} : E \to \mathbb{Z}_+$ *such that for all* $v \in V, x(\delta(v)) \leq b_v$.

Definition 8 (c-vertex cover) *A* $\mathbf{c}$*-vertex cover is an assignment* $\mathbf{y} : V \to \mathbb{Z}_+$ *such that for all edges* $e = (u, v), y_u + y_v \geq c_e$.

Let $\mathbf{b} \in \mathbb{Z}_+^V$ and $\mathbf{c} \in \mathbb{Z}_+^E$. We have

$$\max\{\mathbf{c}^T\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} = \min\{\mathbf{b}^T\mathbf{y} : \mathbf{A}^T\mathbf{y} \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}\}.$$

By the total unimodularity of $\mathbf{A}$, vertices of these polyhedra are integer vectors, so optimal solutions can be taken as integer. (The polyhedron in the dual is unbounded, but the optimum does not change if we constrain the polyhedron for example by $y_v \leq \max c_e$ for all $v \in V$. Then we get a bounded integer polytope and hence there is an integer optimum.) The primal can be interpreted as a maximum $\mathbf{b}$-matching and the dual as a minimum $\mathbf{c}$-vertex cover.

Theorem 9 (Generalized König's Theorem) *For all bipartite graphs with* $\mathbf{b} \in \mathbb{Z}_+^V, \mathbf{c} \in \mathbb{Z}_+^E$*, the Max* $\mathbf{c}$*-weighted* $\mathbf{b}$*-matching is equal to the Min* $\mathbf{b}$*-weighted* $\mathbf{c}$*-vertex covers.*

4 What is the importance of min-max relations?

For example, we know the maximum matching is equal to the minimum vertex cover. The question "Is there a matching of size k?" is obviously in NP. But is it in P?

- If the answer is YES, there is a certificate which can be verified easily - that's why it is in NP.
- If the answer is NO, there is also a certificate, since we can find a vertex cover of size less than k - hence it is in coNP.

Therefore, the problem is in NP $\cap$ coNP which is a good sign that it might be also in P.